

# Quantum theories of $(p, q)$ -forms

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**ABSTRACT:** We describe quantum theories for massless  $(p, q)$ -forms living on Kähler spaces. In particular we consider four different types of quantum theories: two types involve gauge symmetries and two types are simpler theories without gauge invariances. The latter can be seen as building blocks of the former. Their equations of motion can be obtained in a natural way by first-quantizing a spinning particle with a  $U(2)$ -extended supersymmetry on the worldline. The particle system contains four supersymmetric charges, represented quantum mechanically by the Dolbeault operators  $\partial$ ,  $\bar{\partial}$ , and their hermitian conjugates  $\partial^\dagger$ ,  $\bar{\partial}^\dagger$ . After studying how the  $(p, q)$ -form field theories emerge from the particle system, we investigate their one loop effective actions, identify corresponding heat kernel coefficients, and derive exact duality relations. The dualities are seen to include mismatches related to topological indices and analytic torsions, which are computed as  $\text{Tr}(-1)^F$  and  $\text{Tr}(-1)^F F$  in the first quantized supersymmetric nonlinear sigma model for a suitable fermion number operator  $F$ .

**KEYWORDS:** Sigma Models, Duality in Gauge Field Theories

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## 1 Introduction

The aim of this paper is to investigate several quantum theories for  $(p, q)$ -forms living on Kähler spaces and study their duality relations at the quantum level. The most interesting ones contain gauge invariances and generalize in a natural way the theory of differential  $p$ -forms  $C$  with equations of motion of the Maxwell type  $d^\dagger dC = 0$  and gauge invariance  $\delta C = d\Lambda$ . For an arbitrary  $(p, q)$ -form  $A$  they read

$$\left(\partial^\dagger \partial - \bar{\partial} \bar{\partial}^\dagger\right) A = \partial \bar{\partial} \rho \quad (1.1)$$

where  $\partial$  and  $\bar{\partial}$  are the Dolbeault operators with hermitian conjugate  $\partial^\dagger$  and  $\bar{\partial}^\dagger$ , and  $\rho$  is a compensator for the gauge symmetry. For the particular case of a  $(p, d-2-p)$ -form  $\alpha$ , one can eliminate the need of a compensator by suitably modifying the equations to

$$\left(\partial^\dagger \partial - \bar{\partial} \bar{\partial}^\dagger + \partial \bar{\partial} \text{Tr}\right) \alpha = 0 \quad (1.2)$$

where “Tr” takes a trace of the form (with a suitable normalization to be defined later). Simpler theories without gauge invariances can be defined in terms of the Laplace-Beltrami operator  $\Delta$  on  $(p, q)$ -forms  $B$

$$\Delta B = 0 \quad (1.3)$$

and can be seen as building blocks for the quantum theories of the former.

These theories can be obtained from the quantization of the  $U(2)$  spinning particle, originally presented in [1], extended to the  $U(N)$  case in [2], and analyzed in [3] to identify a class of higher spin equations on complex manifolds (see also [4] for a related work on complex interacting higher spin field equations). For  $N \leq 2$  these higher spin equations reduce to equations for ordinary differential  $(p, q)$ -forms on complex manifolds with Kähler metrics. In particular, the case  $N = 1$  describes  $(p, 0)$ -forms and has been discussed in [5]. The ensuing equations have the property that one may allow for an arbitrary coupling to the  $U(1)$  part of the  $U(d)$  holonomy group of the Kähler space. For a special value of this coupling the set of all the  $(p, 0)$ -forms is equivalent to a fermionic Dirac field. In the present paper we wish to develop a similar analysis for the case  $N = 2$ , which is rich enough to provide a description of arbitrary  $(p, q)$ -forms. An analogous treatment for standard  $p$ -forms on real manifolds was given in [6] and [7], where the worldline techniques of [8–10] were used to study the one loop effective actions as function of the background metric. Such a treatment provides a useful guideline for the present analysis as well.

The gauge invariant quantum theories of  $(p, q)$ -forms considered here contain some structural elements present also in the theory of higher spin fields, such as the appearance of compensators to maintain unconstrained gauge invariance [11, 12], as in eq. (1.1), or the emergence of Fronsdal-like equations of motion [14], as in eq. (1.2). A property of the present equations is that they can be defined on arbitrary Kähler spaces. On the contrary, higher spin equations on complex spaces defined by the  $U(N)$  spinning particle require for  $N > 2$  special constraints on the background [3]. This is analogous to the standard theory of higher spin equations described by the  $O(N)$  spinning particle, which for  $N > 2$  requires a conformally flat spacetime [15], including (A)dS as particular cases [16].

Some of the equations presented here, those of the  $\alpha$ -type as in (1.2), were already introduced in [3] using a different basis for the fields, namely a multi-form with two set of  $p$  antisymmetric holomorphic indices, rather than a  $(p, d - 2 - p)$ -form. The basis of  $(p, q)$ -forms is perhaps more intuitive geometrically, and will be used here. Similar equations were also analyzed in [17] by making use of the “detour” construction described in [18–20]. All of these first quantized pictures make use of supersymmetric quantum mechanics [21, 22]. In particular, one needs a supersymmetric quantum mechanics defined by a one-dimensional nonlinear sigma model with Kähler manifolds as target space and four supercharges that give rise to the Dolbeault operators  $\partial, \bar{\partial}$  and their hermitian conjugates  $\partial^\dagger, \bar{\partial}^\dagger$ . This model has been used in [23] to give a supersymmetric proof of the index theorem for the Dolbeault complex (the index of  $\bar{\partial}$  on Kähler manifolds), and its lagrangian reviewed for example in [24]. For our purposes the supersymmetry algebra has to be gauged suitably to provide the spinning particle actions that implement the worldline description of the various  $(p, q)$ -form field theories. Canonical quantization produces the field equations corresponding to the particular model that is selected by the charges that one wishes to gauge. Path integral quantization provides then a simple representation of the corresponding one loop effective action, which we use for computing the first few heat kernel coefficients. Dual descriptions are easily identified in this first quantized picture. They include topological mismatches that are related to topological indices and analytic torsions, which can be obtained in the

supersymmetric quantum mechanics as Witten indices of the form  $\text{Tr}(-1)^F$  for a suitable fermion number operator  $F$  [21], and indices of the type  $\text{Tr}(-1)^F F$ , originally introduced in [25] for two dimensional supersymmetric field theories, respectively.

We start in section 2 with canonical quantization to obtain the equations of motion for the  $(p, q)$ -forms in flat complex space from the spinning particles. The susy algebra contains  $U(2)$  as maximal R-symmetry group. We gauge the full  $U(2)$  group to obtain what we call the  $\alpha$  and  $\beta$  models. Gauging a  $U(1) \times U(1)$  subgroup produces instead the A and B models, that can be defined for arbitrary  $(p, q)$ -forms. Gauging the supercharges produces field theories with gauge invariances, the A and  $\alpha$  models. Keeping the supercharges ungauged gives rise to simpler models without gauge invariances, the B and  $\beta$  models. The hamiltonian is always gauged to obtain from first quantization the standard proper time representation of propagators and effective actions. In section 3 we extend the previous results to complex spaces with an arbitrary Kähler metric, treated as a background field. In section 4 we quantize the particle models on a circle to obtain one-loop effective actions. We compute the first few heat kernel coefficients for all of the models discussed previously using a worldline path integral. In sections 5 we analyze systematically duality relations, and show how a variety of topological mismatches arise for the A-type models. We end up with our conclusions in section 6, and include three appendices to provide conventions, expressions for the worldline propagators, and a list of topological quantities entering the duality relations.

## 2 Canonical quantization and equations of motion in flat space

We are aiming at describing four different quantum theories of differential forms by means of appropriate spinning particle models. The ground on which all these models are constructed is the  $N = 2$  extended complex superalgebra in one dimension. The different versions are obtained by gauging suitable subgroups of the whole superalgebra.

We begin by considering such spinning particle models in flat complex space  $\mathbb{C}^d$  (see Appendix A for notations and conventions). The particle is described by complex coordinates  $x^\mu(t), \bar{x}^{\bar{\mu}}(t)$ ,  $\mu = 1, \dots, d$ , and carries additional degrees of freedom associated to their fermionic superpartners  $\psi_i^\mu(t), \bar{\psi}_\mu^i(t)$ ,  $i = 1, 2$ , belonging to the (anti-)fundamental representation of the  $U(2)$  R-symmetry group. Spacetime indices are lowered and raised with the flat metric  $\delta_{\mu\bar{\nu}}$  and its inverse, so that we will often use  $\bar{\psi}^{i\bar{\mu}} = \delta^{\nu\bar{\mu}} \bar{\psi}_\nu^i$  and  $\psi_{i\bar{\mu}} = \delta_{\nu\bar{\mu}} \psi_i^\nu$ . With the above ingredients, the ungauged model is described by the phase-space action

$$S = \int_0^1 dt \left[ p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i \bar{\psi}_\mu^i \dot{\psi}_i^\mu - p_\mu \bar{p}^\mu \right], \quad (2.1)$$

which encodes the motion of a free particle with a pseudoclassical spin associated to the Grassmann coordinates. The corresponding conserved charges

$$H = p_\mu \bar{p}^\mu, \quad Q_i = \psi_i^\mu p_\mu, \quad \bar{Q}^i = \bar{\psi}^{i\bar{\mu}} \bar{p}_{\bar{\mu}}, \quad J_i^j = \psi_i^\mu \bar{\psi}_\mu^j \quad (2.2)$$

generate a  $U(2)$ -extended supersymmetry algebra on the wordline. Indeed, by using the commutation relations that follow from the classical Poisson bracket algebra

$$[x^\mu, p_\nu] = i\delta_\nu^\mu, \quad [\bar{x}^{\bar{\mu}}, \bar{p}_{\bar{\nu}}] = i\delta_{\bar{\nu}}^{\bar{\mu}}, \quad \{\psi_i^\mu, \bar{\psi}_\nu^j\} = \delta_\nu^\mu \delta_i^j, \quad (2.3)$$

it is easy to check that the above conserved charges satisfy the quantum commutation relations

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= \delta_i^j H, \\ [J_i^j, Q_k] &= \delta_k^j Q_i, \quad [J_i^j, \bar{Q}^k] = -\delta_i^k \bar{Q}^j, \\ [J_i^j, J_k^l] &= \delta_k^j J_i^l - \delta_i^l J_k^j, \end{aligned} \quad (2.4)$$

while other independent graded-commutators vanish. Note that ordering ambiguities are only present in  $J_i^j$ , for which we choose the graded-symmetric ordering *i.e.*

$$J_i^j = \frac{1}{2}(\psi_i^\mu \bar{\psi}_\mu^j - \bar{\psi}_\mu^j \psi_i^\mu) = \psi_i^\mu \bar{\psi}_\mu^j - \frac{d}{2} \delta_i^j \quad (2.5)$$

that is naturally reproduced by the fermionic path integral of section 4. Choosing the Hilbert space basis of  $(x^\mu, \bar{x}^{\bar{\mu}}, \psi_1^\mu, \bar{\psi}^{\bar{\mu}2})$ -eigenstates, the corresponding momenta are realized as

$$p_\mu \sim -i\partial_\mu, \quad \bar{p}_{\bar{\mu}} \sim -i\partial_{\bar{\mu}}, \quad \bar{\psi}_\mu^1 \sim \frac{\partial}{\partial \psi_1^\mu}, \quad \psi_{2\bar{\mu}} \sim \frac{\partial}{\partial \bar{\psi}^{2\bar{\mu}}}. \quad (2.6)$$

The states are therefore represented by functions  $F(x, \bar{x}, \psi_1, \bar{\psi}^2) = \langle x, \bar{x}, \psi_1, \bar{\psi}^2 | F \rangle$ , that, due to the Grassmannian nature of the  $\psi$  variables, admit the finite Taylor expansion

$$F(x, \bar{x}, \psi_1, \bar{\psi}^2) = \sum_{m,n=0}^d F_{\mu_1 \dots \mu_m, \bar{\nu}_1 \dots \bar{\nu}_n}(x, \bar{x}) \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}^{2\bar{\nu}_1} \dots \bar{\psi}^{2\bar{\nu}_n}, \quad (2.7)$$

where the coefficients  $F_{\mu_1 \dots \mu_m, \bar{\nu}_1 \dots \bar{\nu}_n}(x, \bar{x})$  are components of complex  $(m, n)$ -forms with  $m$  antisymmetric holomorphic and  $n$  antisymmetric anti-holomorphic indices. In the following, we shall sometimes use the shorthand notation  $F_{\mu[m], \bar{\nu}[n]}(x, \bar{x})$ , where  $m, n \in \mathbb{N}$  within square brackets denote the number of antisymmetrized indices of each kind. Identifying  $\psi_1^\mu$  with the holomorphic differential  $dx^\mu$  and  $\bar{\psi}^{2\bar{\mu}}$  with the anti-holomorphic one  $d\bar{x}^{\bar{\mu}}$ , we can identify the various terms in the sum with  $(m, n)$ -forms

$$F_{(m,n)}(x, \bar{x}) := F_{\mu[m], \bar{\nu}[n]}(x, \bar{x}) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} \wedge d\bar{x}^{\bar{\nu}_1} \wedge \dots \wedge d\bar{x}^{\bar{\nu}_n}. \quad (2.8)$$

Wedge products will be understood in the following whenever differential forms are juxtaposed. On the above Hilbert-space states the quantized conserved charges are represented by differential operators. In particular, the supercharges  $Q_1$  and  $\bar{Q}^2$  act, up to an imaginary factor, as the Dolbeault operators  $\partial$  and  $\bar{\partial}$

$$iQ_1 \sim \partial = dx^\mu \partial_\mu, \quad i\bar{Q}^2 \sim \bar{\partial} = d\bar{x}^{\bar{\mu}} \partial_{\bar{\mu}}, \quad (2.9)$$

on  $(m, n)$ -forms. Similarly,

$$i\bar{Q}^1 \sim -\partial^\dagger = \delta^{\mu\bar{\nu}} \partial_{\bar{\nu}} \frac{\partial}{\partial (dx^\mu)}, \quad iQ_2 \sim -\bar{\partial}^\dagger = \delta^{\mu\bar{\nu}} \partial_\mu \frac{\partial}{\partial (d\bar{x}^{\bar{\nu}})}, \quad (2.10)$$

correspond, up to a sign, to the adjoint Dolbeault operators  $\partial^\dagger$  and  $\bar{\partial}^\dagger$  (see Appendix A) that act on  $F_{(m,n)}$  by taking a divergence in the holomorphic or anti-holomorphic indices, respectively. The hamiltonian is realized as the laplacian operator

$$H \sim -\Delta := (\partial\partial^\dagger + \partial^\dagger\partial) = (\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}) = -\delta^{\mu\bar{\nu}}\partial_{\bar{\nu}}\partial_\mu = -\frac{1}{2}\square, \quad (2.11)$$

where we denote with  $\square := \partial^M\partial_M$  the laplacian operator in  $\mathbb{R}^{2d}$  with its standard normalization. Finally, the diagonal  $R$ -symmetry generators  $J_1^1$  and  $J_2^2$  essentially count the holomorphic and antiholomorphic degree of a  $(m,n)$ -form,

$$J_1^1 = N - \frac{d}{2}, \quad N \sim dx^\mu \frac{\partial}{\partial(dx^\mu)} \quad (2.12)$$

$$J_2^2 = -\bar{N} + \frac{d}{2}, \quad \bar{N} \sim d\bar{x}^{\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\nu}})}, \quad (2.13)$$

respectively ( $N$  and  $\bar{N}$  correspond to the fermionic number operators in the holomorphic and anti-holomorphic sector); and the off-diagonal  $R$ -symmetry generators  $J_1^2$  and  $J_2^1$  correspond to inserting the metric and to taking a trace, respectively

$$J_1^2 \sim \delta \wedge = \delta_{\mu\bar{\nu}} dx^\mu d\bar{x}^{\bar{\nu}}, \quad J_2^1 \sim -\text{Tr} = -\delta^{\mu\bar{\nu}} \frac{\partial^2}{\partial(dx^\mu)\partial(d\bar{x}^{\bar{\nu}})}. \quad (2.14)$$

We shall now proceed by gauging different subalgebras of the whole rigid supersymmetry algebra (2.4) generated by  $C_I = (H, Q_i, \bar{Q}^i, J_i^j)$ . The models we shall focus on are:

- $A$ -model, obtained by gauging the hamiltonian  $H$ , the supercharges  $Q_i, \bar{Q}^i$  and the  $U(1) \times U(1)$  subgroup of the  $R$ -symmetry group  $U(2)$ , generated by  $J_1^1$  and  $J_2^2$ . We denote the gauged generators by  $C_I^A = (H, Q_i, \bar{Q}^i, J_1^1, J_2^2)$ ;
- $\alpha$ -model, obtained by gauging the whole  $U(2)$ -extended supersymmetry algebra (2.4), *i.e.*  $C_I^\alpha = (H, Q_i, \bar{Q}^i, J_i^j)$ ;
- $B$ -model, obtained by leaving the supersymmetry rigid and gauging the hamiltonian  $H$  and the  $U(1) \times U(1)$   $R$ -symmetry subgroup,  $C_I^B = (H, J_1^1, J_2^2)$ ;
- $\beta$ -model, obtained by gauging the hamiltonian and the whole  $R$ -symmetry algebra, *i.e.* all generators (2.2) except for the supercharges, thus  $C_I^\beta = (H, J_i^j)$ .

The corresponding gauged worldline actions in phase space are obtained by coupling the correct subset of Noether charges  $C_I$  to worldline gauge fields  $G^I = (e, i\bar{\chi}^i, i\chi_i, a_j^i)$ , where  $e$  is the einbein,  $\chi_i, \bar{\chi}^i$  are complex gravitinos, and  $a_j^i$  is a worldline  $U(2)$  gauge field. By denoting the four models collectively as  $\mathcal{A} = (A, \alpha, B, \beta)$ , such actions read as

$$S_{\mathcal{A}} = \int_0^1 dt \left[ p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_\mu^i \dot{\psi}_i^\mu - C_I^{\mathcal{A}} G^I \right] + S_{\mathcal{A}}^{CS}, \quad (2.15)$$

where  $C_I^{\mathcal{A}}$  denotes the proper subset of charges that are gauged, while  $S_{\mathcal{A}}^{CS}$  is the allowed Chern-Simons term for the  $a$  gauge field. When the whole  $U(2)$  symmetry is gauged, only one Chern-Simons coupling  $s$  is allowed

$$S_{\alpha}^{CS} = S_{\beta}^{CS} = s \int_0^1 dt a_i^i, \quad (2.16)$$

while, if only the subgroup  $U(1) \times U(1)$  has to be gauged, one can add two independent couplings to the two  $U(1)$  factors,  $s_1$  and  $s_2$ ,

$$S_A^{CS} = S_B^{CS} = \int_0^1 dt [s_1 a_1^1 + s_2 a_2^2]. \quad (2.17)$$

The classical equations of motion for the gauge fields  $G^I$  constrain the Noether charges to vanish. Quantum-mechanically, this translates into the selection of the physical Hilbert space, which is obtained by requiring that the symmetry generators annihilate physical states,

$$|F^{\mathcal{A}}\rangle \in \mathcal{H}_{phys}^{\mathcal{A}} \iff T_I^{\mathcal{A}} |F^{\mathcal{A}}\rangle = 0, \quad \forall I, \quad (2.18)$$

where we have defined (including the ordering issue discussed in eq. (2.5))

$$T_I^{\mathcal{A}} := -\frac{\delta S^{\mathcal{A}}}{\delta G^I} = C_I^{\mathcal{A}} - \frac{\delta S_{CS}^{\mathcal{A}}}{\delta G^I}. \quad (2.19)$$

Clearly, only the  $J$ -constraints are affected by the shift due to the Chern-Simons term, and take the form  $T_J^{\mathcal{A}} = J_i^j - s\delta_i^j$  for  $\mathcal{A} = \alpha, \beta$  and  $T_J^{\mathcal{A}} = (J_1^1 - s_1, J_2^2 - s_2)$  for  $\mathcal{A} = A, B$ .

We now turn to examine the constraints in the different models and the resulting requirements they put on physical states.

- *$\alpha$ -model*

We impose the diagonal  $J$ -constraints first. They take the form

$$\begin{aligned} (J_1^1 - s)|F\rangle = 0 &\longrightarrow (N - m) \sum_{k,l=0}^d F_{(k,l)} = 0, \\ (J_2^2 - s)|F\rangle = 0 &\longrightarrow (-\bar{N} + d - m) \sum_{k,l=0}^d F_{(k,l)} = 0 \quad m := s + \frac{d}{2} \in \mathbb{N}. \end{aligned} \quad (2.20)$$

The restriction that  $m$  be a natural number follows from the fact that  $N$  and  $\bar{N}$  count the number of holomorphic and anti-holomorphic indices, respectively. This requirement in turn fixes the possible quantized values of the Chern-Simons coupling  $s$ . In summary, the  $J_i^i$ -constraints at fixed  $i = 1, 2$  select, out of the various terms in (2.7), the  $(p+1, d-p-1)$ -form

$$F_{(p+1, d-p-1)} = F_{\mu[p+1], \bar{\nu}[d-p-1]} dx^{\mu_1} \dots dx^{\mu_{p+1}} d\bar{x}^{\bar{\nu}_1} \dots d\bar{x}^{\bar{\nu}_{d-p-1}}, \quad (2.21)$$

as physical particle states in the  $\alpha$ -model, where we have set for convenience  $m \equiv p+1$ . Imposing now  $J_2^1$  on (2.21) corresponds to declaring that

$$\text{Tr}(F_{(p+1, d-1-p)}) = 0 . \quad (2.22)$$

While independent of  $J_2^1$  at the level of the commutation relations (2.4), the  $J_1^2$  constraint is nonetheless automatically satisfied if (2.22) holds, as one can show by letting  $J_1^2$  act on (2.21), using (2.14) and the identity

$$dx^{\mu_1} \dots dx^{\mu_k} = \frac{1}{k!(d-k)!} \epsilon^{\mu_1 \dots \mu_k \nu_{k+1} \dots \nu_d} \epsilon_{\rho_1 \dots \rho_k \nu_{k+1} \dots \nu_d} dx^{\rho_1} \dots dx^{\rho_k} , \quad (2.23)$$

along with its anti-holomorphic analogue. Let us stress that on a general Kähler manifold the (anti-)holomorphic volume form is not globally defined, unless the manifold is Calabi-Yau, *i.e.* it has  $SU(d)$  holonomy or, equivalently, admits a Ricci-flat metric. If this is not the case, the equivalence between  $J_2^1$  and  $J_1^2$  constraints only holds locally. We shall return to these issues in the forthcoming sections dealing with curved backgrounds.

As for the supercharges, it is immediate to see from the second relation in (2.4) that, once the  $J$ -constraints have been imposed, there are only two independent  $(Q, \bar{Q})$ -constraints. We choose them to be  $Q_1$  and  $\bar{Q}^2$ . From the realizations (2.9) and (2.10) the conditions  $Q_1|F\rangle = 0 = \bar{Q}^2|F\rangle$  correspond to (anti-)holomorphic integrability equations

$$\partial F_{(p+1, d-1-p)} = 0 , \quad \bar{\partial} F_{(p+1, d-1-p)} = 0 , \quad (2.24)$$

which are solved by

$$F_{(p+1, d-1-p)} = \partial \bar{\partial} \alpha_{(p, d-2-p)} , \quad (2.25)$$

where  $\alpha_{(p, d-2-p)}$  is a potential represented by a complex  $(p, d-2-p)$ -form. With such a position, the above equations become identities (Bianchi identities). The spinning particle physical states therefore can be interpreted as curvatures for a complex gauge field  $\alpha_{(p, d-2-p)}$ . Eq. (2.22) now becomes a second-order wave equation

$$\text{Tr}(\partial \bar{\partial} \alpha_{(p, d-2-p)}) = \left( \Delta + \partial \partial^\dagger + \bar{\partial} \bar{\partial}^\dagger - \partial \bar{\partial} \text{Tr} \right) \alpha_{(p, d-2-p)} = 0 , \quad (2.26)$$

which just corresponds to a complexified version of Fronsdal's equations [3, 14]. Indeed, in components it reads

$$\begin{aligned} & \partial_\rho \bar{\partial}^\rho \alpha_{\mu[p], \bar{\nu}[d-2-p]} - p \partial_{[\mu_1} \bar{\partial}^\rho \alpha_{\rho|\mu_2 \dots \mu_p], \bar{\nu}[d-2-p]} - (d-2-p) \partial_{[\bar{\nu}_1} \bar{\partial}^\rho \alpha_{\mu[p], \bar{\rho}|\bar{\nu}_2 \dots \bar{\nu}_{d-2-p}]} \\ & + p(d-2-p) \partial_{[\mu_1} \partial_{[\bar{\nu}_1} \alpha_{\mu_2 \dots \mu_p], \bar{\sigma}|\bar{\nu}_2 \dots \bar{\nu}_{d-2-p}]} = 0 , \end{aligned} \quad (2.27)$$

where we denoted weighted anti-symmetrization with square brackets. Clearly, the curvatures and the equations of motion (2.26) are invariant under the gauge transformations

$$\delta \alpha_{(p, d-2-p)} = \partial \Lambda_{(p-1, d-2-p)}^1 + \bar{\partial} \Lambda_{(p, d-3-p)}^2 . \quad (2.28)$$



As we mentioned in the introduction, the  $\alpha$ -form system discussed above is physically equivalent to the bi-forms described in [3] for  $N = 2$ , since they come from the very same quantum mechanical sigma model. In the former work it has been used a Hilbert space basis of  $(\psi_1, \psi_2)$  eigenstates, instead of the present basis of  $(\psi_1, \bar{\psi}^2)$  eigenstates. This leads to different geometric realizations of the same quantum mechanical operators, the curvatures in the two realizations being related by a holomorphic Hodge duality as

$$F_{\mu[m], \bar{\nu}_1 \dots \bar{\nu}_{d-m}}^{\text{new}} \propto F_{\mu[m], \nu_1 \dots \nu_m}^{\text{old}} \epsilon^{\nu_1 \dots \nu_m} \bar{\nu}_1 \dots \bar{\nu}_{d-m} ,$$

and gauge fields are introduced by solving different integrability equations that give rise to different Bianchi identities. Hence, gauge potentials in the two realizations are not related in a local way, and the corresponding field theories are not manifestly equivalent at the level of equations of motion.

- *A-model*

The difference with the  $\alpha$ -model consists in the absence of the non-diagonal  $J$ -constraints. As a consequence, as explained in (2.17), here we have two independent Chern-Simons couplings, one for each  $U(1)$ . In other words, gauging  $(J_1^1, J_2^2)$  enforces the requirements

$$\begin{aligned} (J_1^1 - s_1)|F\rangle = 0 &\longrightarrow (N - m) \sum_{k,l=0}^d F_{(k,l)} = 0, \quad m := s_1 + \frac{d}{2} \in \mathbb{N}, \\ (J_2^2 - s_2)|F\rangle = 0 &\longrightarrow (-\bar{N} + n) \sum_{k,l=0}^d F_{(k,l)} = 0, \quad n := -s_2 + \frac{d}{2} \in \mathbb{N}, \end{aligned} \quad (2.29)$$

that select physical states represented by  $(p+1, q+1)$ -forms

$$F_{(p+1, q+1)} = F_{\mu[p+1], \bar{\nu}[q+1]} dx^{\mu_1} \dots dx^{\mu_{p+1}} d\bar{x}^{\bar{\nu}_1} \dots d\bar{x}^{\bar{\nu}_{q+1}}, \quad (2.30)$$

where we have set  $m \equiv p+1$ ,  $n \equiv q+1$ . Now that  $J_2^1$  and  $J_1^2$  are no longer imposed, all supersymmetry generators give rise to independent constraints. As before,  $Q_1$  and  $\bar{Q}^2$  enforce the integrability equations

$$\partial F_{(p+1, q+1)} = 0, \quad \bar{\partial} F_{(p+1, q+1)} = 0, \quad (2.31)$$

that are solved in terms of a  $(p, q)$ -form potential  $A_{(p, q)}$ ,

$$F_{(p+1, q+1)} = \partial \bar{\partial} A_{(p, q)}. \quad (2.32)$$

The remaining constraints  $\bar{Q}^1|F\rangle = 0 = Q_2|F\rangle$  translate into

$$\partial^\dagger(\partial \bar{\partial} A_{(p, q)}) = \bar{\partial}(-\Delta A_{(p, q)} - \partial \partial^\dagger A_{(p, q)}) = 0, \quad (2.33)$$

$$\bar{\partial}^\dagger(\partial \bar{\partial} A_{(p, q)}) = \partial(\Delta A_{(p, q)} + \bar{\partial} \bar{\partial}^\dagger A_{(p, q)}) = 0, \quad (2.34)$$

from which, taking into account that  $\partial^2 = 0 = \bar{\partial}^2$ , it follows that  $(\Delta + \partial \partial^\dagger + \bar{\partial} \bar{\partial}^\dagger) A_{(p, q)} \in \text{Ker}(\partial, \bar{\partial})$ . In other words, the field equations

$$(\Delta + \partial \partial^\dagger + \bar{\partial} \bar{\partial}^\dagger) A_{(p, q)} = \partial \bar{\partial} \rho_{(p-1, q-1)}, \quad (2.35)$$

where  $\rho_{(p-1, q-1)}$  is a compensator field, are satisfied by the  $(p, q)$ -form potential  $A_{(p, q)}$  characterizing the physical spinning particle states in the  $A$ -model via (2.32). Eq. (2.35) is a complex version of the Francia-Sagnotti equations adapted to a  $(p, q)$ -form potential [11, 12]. A geometrical derivation of similar Maxwell-like equations with compensators leading to expressions close in form to (2.35) was given in [13]. The curvature (2.32) is left invariant by the gauge transformations

$$\delta A_{(p, q)} = \partial \Lambda_{(p-1, q)}^1 + \bar{\partial} \Lambda_{(p, q-1)}^2 . \quad (2.36)$$

This transformation preserves the field equations (2.35) provided that it is accompanied by the gauge variation of the compensators

$$\delta \rho_{(p-1, q-1)} = \bar{\partial}^\dagger \Lambda_{(p-1, q)}^1 - \partial^\dagger \Lambda_{(p, q-1)}^2 , \quad (2.37)$$

as it can be proved by direct substitution. It is possible, by means of (2.37), to gauge fix the compensators to zero, recovering the field equations in the homogeneous form  $(\Delta + \partial \partial^\dagger + \bar{\partial} \bar{\partial}^\dagger) A_{(p, q)} = 0$ . The residual gauge symmetry parameters are then forced to satisfy  $\bar{\partial}^\dagger \Lambda_{(p-1, q)}^1 - \partial^\dagger \Lambda_{(p, q-1)}^2 = 0$ , analogously to the case studied in [3]. Moreover, by taking divergencies of the gauge fixed field equations, one finds that the  $(p, q)$ -form has to be doubly divergenceless:  $\partial^\dagger \bar{\partial}^\dagger A_{(p, q)} = 0$ .

- *$\beta$ -model*

In this case the hamiltonian and the whole  $U(2)$   $R$ -symmetry algebra are gauged, while the supersymmetries are not. This means that the  $J$ -constraints (2.20)-(2.22) are all imposed, but there are no integrability equations to be solved in terms of potentials, nor there are Fronsdal-like equations for the latter. This implies that the  $(m, n)$ -forms  $F_{(m, n)}$  that enter the expansion (2.7) have no potential and no associated gauge symmetry. For this reason, we shall here set  $m \equiv p$  for a direct comparison with the previous models and write the  $(m, n)$ -forms selected by (2.20) as  $\beta_{(p, d-p)}$ . Therefore, the physical states in this model are represented by traceless  $(p, d-p)$ -forms  $\beta_{(p, d-p)}$  satisfying a Klein-Gordon equation

$$\Delta \beta_{(p, d-p)} = 0 , \quad \text{Tr}(\beta_{(p, d-p)}) = 0 . \quad (2.38)$$

- *$B$ -model*

Again the supersymmetries remain rigid and the only  $R$ -symmetry constraints that are imposed are (2.29). As a result, in the  $B$ -model the physical states are represented by  $(p, q)$ -forms  $B_{(p, q)}$  satisfying the Klein-Gordon equation

$$\Delta B_{(p, q)} = 0 \quad (2.39)$$

with no associated gauge symmetries.

Though derived in flat space, all these equations, suitably covariantized, hold on a generic Kähler manifold, as the analysis in the following section will show.

### 3 $(p, q)$ -forms on Kähler manifolds

We shall now analyze the coupling of the various  $(p, q)$ -forms considered above to an arbitrary background Kähler metric  $g_{\mu\bar{\nu}}(x, \bar{x}) = g_{\bar{\nu}\mu}(x, \bar{x})$ . The Grassmann variables  $(\psi_i^\mu(t), \bar{\psi}_\mu^i(t))$  now transform as contravariant or covariant vectors under holomorphic change of coordinates, and we stress that in curved space we choose to define the  $\bar{\psi}$ 's with an holomorphic lower vector index, in order to avoid position dependent anti-commutation relations<sup>1</sup>. Accordingly, appropriate covariantizations of the supersymmetry charges need to be constructed, such that their anticommutation relations produce the correct supersymmetry algebra and related hamiltonian. Recalling that on Kähler manifolds the only non-vanishing components of the Christoffel connection are the purely holomorphic  $\Gamma_{\mu\nu}^\lambda$  and purely anti-holomorphic ones  $\Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}}$ , one can substitute the momenta  $(p_\mu, \bar{p}_{\bar{\mu}})$  with the covariant momenta  $(\pi_\mu, \bar{\pi}_{\bar{\mu}})$  defined as

$$\pi_\mu = p_\mu + i \Gamma_{\mu\nu}^\lambda \psi_i^\nu \bar{\psi}_\lambda^i, \quad \bar{\pi}_{\bar{\mu}} = \bar{p}_{\bar{\mu}}, \quad (3.1)$$

whose Poisson bracket is proportional to the Riemann curvature tensor (see Appendix A for our curvature conventions)

$$\{\pi_\mu, \bar{\pi}_{\bar{\nu}}\}_{PB} = i R_{\mu\bar{\nu}}^\lambda \psi_i^\nu \bar{\psi}_\lambda^i. \quad (3.2)$$

Therefore, the classical covariantized supercharges read

$$Q_i = \psi_i^\mu \pi_\mu, \quad \bar{Q}^i = \bar{\psi}_\mu^i g^{\mu\bar{\nu}} \bar{p}_{\bar{\nu}}, \quad (3.3)$$

and their Poisson bracket leads to the classical hamiltonian

$$\begin{aligned} H_{cl} &= g^{\mu\bar{\nu}} \bar{\pi}_{\bar{\nu}} \pi_\mu - \frac{1}{2} R_{\mu\nu}^\lambda \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \\ &= g^{\mu\bar{\nu}} \bar{p}_{\bar{\nu}} (p_\mu + i \Gamma_{\mu\nu}^\lambda \psi_i^\nu \bar{\psi}_\lambda^i) - \frac{1}{2} R_{\mu\nu}^\lambda \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j. \end{aligned} \quad (3.4)$$

The classical phase space actions for the various models can now be written in terms of the covariant momenta as

$$S_{\mathcal{A}} = \int_0^1 dt \left[ p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i \bar{\psi}_\mu^i \dot{\psi}_i^\mu - \mathcal{C}_I^A G^I \right] + S_{\mathcal{A}}^{CS} \quad (3.5)$$

where  $\mathcal{C}_I^A$  differs from  $C_I^A$  of the previous section for the substitution of the flat space constraints with the covariantized ones (3.3), (3.4).

At the quantum level an ordering prescription is again needed for the  $U(d)$  generators  $\psi_i^\mu \bar{\psi}_\lambda^i$  appearing in the covariant momenta (3.1). We therefore set

$$\pi_\mu = p_\mu + i \Gamma_{\mu\nu}^\lambda M_\lambda^\nu, \quad \bar{\pi}_{\bar{\mu}} = \bar{p}_{\bar{\mu}}, \quad (3.6)$$

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<sup>1</sup>These variables correspond to Darboux coordinates, that make the graded symplectic form assume a canonical expression with constant components. By canonical quantization these coordinates acquire then position independent anti-commutation relations.

where the  $U(d)$  “Lorentz” generators

$$M_\nu^\mu := \frac{1}{2} [\psi_i^\mu, \bar{\psi}_\nu^i] = \psi_i^\mu \bar{\psi}_\nu^i - \delta_\nu^\mu \quad (3.7)$$

(see also Eq. (A.13)) are here defined with the graded-symmetric ordering prescription, and their commutator reads

$$[\pi_\mu, \bar{\pi}_{\bar{\nu}}] = -R_{\mu\bar{\nu}}{}^\lambda{}_\nu M_\lambda^\nu. \quad (3.8)$$

The covariant momenta are hermitian conjugate to each other with respect to the inner product

$$\langle \chi | \phi \rangle = \int d^d x d^d \bar{x} g(x, \bar{x}) d^d \psi d^d \bar{\psi} e^{\bar{\psi}_\mu^\mu \psi_i^\mu} \overline{\chi(x, \bar{x}, \psi_1, \bar{\psi}^2)} \phi(x, \bar{x}, \psi_1, \bar{\psi}^2) \quad (3.9)$$

where  $g = \det g_{\mu\bar{\nu}}$ . Note that with this inner product  $(\psi_i^\mu)^\dagger = \bar{\psi}_\nu^i g^{\mu\bar{\nu}}$ , and the hermiticity property of the momentum reads:  $p_\mu^\dagger = \bar{p}_{\bar{\mu}} + i g_{\lambda\bar{\lambda}} g^{\nu\bar{\nu}} \Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}} M_\nu^\lambda$ . The supercharges then take the form

$$Q_i = \psi_i^\mu g^{1/2} \pi_\mu g^{-1/2}, \quad (3.10)$$

$$\bar{Q}^i = \bar{\psi}_\mu^i g^{\mu\bar{\nu}} g^{1/2} \bar{\pi}_{\bar{\nu}} g^{-1/2}, \quad (3.11)$$

where the wrappings  $g^{1/2} \dots g^{-1/2}$  ensure their correct hermiticity properties [3, 5] since, with respect to the inner product (3.9), one has  $Q_i^\dagger = \bar{Q}^i$ . The covariantized supercharges satisfy the anti-commutation relation

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= g^{1/2} \{ \psi_i^\mu \pi_\mu, \bar{\psi}_\mu^j g^{\mu\bar{\nu}} \bar{\pi}_{\bar{\nu}} \} g^{-1/2} \\ &= \delta_i^j g^{1/2} g^{\mu\bar{\nu}} \bar{\pi}_{\bar{\nu}} \pi_\mu g^{-1/2} - \psi_i^\mu \bar{\psi}^{j\bar{\nu}} R_{\mu\bar{\nu}\sigma\bar{\lambda}} M^{\sigma\bar{\lambda}} \\ &= \delta_i^j \left( g^{1/2} g^{\mu\bar{\nu}} \bar{\pi}_{\bar{\nu}} \pi_\mu g^{-1/2} - \frac{1}{2} R_{\mu\bar{\nu}\sigma\bar{\lambda}} M^{\mu\bar{\nu}} M^{\sigma\bar{\lambda}} + \frac{1}{2} R_{\mu\bar{\nu}} M^{\mu\bar{\nu}} \right) \\ &= \delta_i^j H, \end{aligned} \quad (3.12)$$

where the fact that  $N = 2$  is crucial for the next-to-last equality, and where we have defined

$$H := H_0 - \frac{1}{2} R_{\mu\bar{\nu}\sigma\bar{\lambda}} M^{\mu\bar{\nu}} M^{\sigma\bar{\lambda}} \quad (3.13)$$

$$H_0 := \frac{1}{2} g^{1/2} g^{\mu\bar{\nu}} (\bar{\pi}_{\bar{\nu}} \pi_\mu + \pi_\mu \bar{\pi}_{\bar{\nu}}) g^{-1/2}. \quad (3.14)$$

The other commutators (2.4) remain the same as in the flat case. One therefore achieves closure of the constraint algebra with the complete hamiltonian  $H$ , containing the symmetric, minimally-covariantized piece  $H_0$  and a non-minimal contribution proportional to the Riemann curvature. A point worth stressing is that the closure of the  $(Q_i, \bar{Q}^j)$  algebra on  $H$  forbids an extra coupling to the  $U(1)$  part of the Kähler connection, differently from the case of  $(p, 0)$ -forms treated in [5], for which an arbitrary  $U(1)$  charge is allowed. With this restriction one sees that performing (anti-)holomorphic Hodge dualities with the chiral

epsilon tensor  $e \epsilon_{\mu_1 \dots \mu_d}$  would lead us outside the class of models described by our spinning particle, since unwanted couplings to the  $U(1)$  part of the holonomy would be introduced<sup>2</sup>.

Having presented the covariant model on a Kähler manifold, it is now useful to spend some words on the geometric realization of the operators and the Hilbert space states. Since we are expanding the states in powers of  $\psi_1^\mu$  and  $\bar{\psi}_\mu^2$ , our tensors contain only holomorphic indices, and take the form

$$F(x, \bar{x}, \psi_1, \bar{\psi}^2) = \sum_{m,n=0}^d F_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}(x, \bar{x}) \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}_{\nu_1}^2 \dots \bar{\psi}_{\nu_n}^2. \quad (3.15)$$

These tensors are of course equivalent to the  $(m, n)$ -forms by raising the antiholomorphic indices with the Kähler metric

$$F_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} = g^{\nu_1 \bar{\nu}_1} \dots g^{\nu_n \bar{\nu}_n} F_{\mu_1 \dots \mu_m, \bar{\nu}_1 \dots \bar{\nu}_n}. \quad (3.16)$$

The covariant momenta (3.6) acts on the tensors in (3.15) as covariant derivatives:

$$g^{1/2} \pi_\mu g^{-1/2} \sim -i \nabla_\mu, \quad g^{1/2} \bar{\pi}_{\bar{\mu}} g^{-1/2} \sim -i \partial_{\bar{\mu}} = -i \nabla_{\bar{\mu}}, \quad (3.17)$$

and only when the operators are expressed as covariant geometric objects one is free to pass through the  $g$  factors in (3.16), and let them act on the true  $(m, n)$ -form fields. Hence, once the covariant action of geometric operators is understood, it is possible to identify the  $\bar{\psi}^2$ 's as anti-holomorphic form basis via  $g^{\mu\bar{\nu}} \bar{\psi}_\mu^2 \sim d\bar{x}^{\bar{\nu}}$ , and the conjugate momenta as formal derivatives thereof:  $\bar{\psi}_2^\mu \sim g^{\mu\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\nu}})}$ . To make more explicit how this is realized, we write the action of the covariant momentum  $\bar{\pi}_{\bar{\mu}} = \bar{p}_{\bar{\mu}}$  on a state in (3.15)

$$\begin{aligned} & i g^{1/2} \bar{\pi}_{\bar{\mu}} g^{-1/2} F_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}_{\nu_1}^2 \dots \bar{\psi}_{\nu_n}^2 \\ &= \left[ \partial_{\bar{\mu}} F_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} \right] \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}_{\nu_1}^2 \dots \bar{\psi}_{\nu_n}^2 \\ &= \left[ \nabla_{\bar{\mu}} F_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} \right] \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}_{\nu_1}^2 \dots \bar{\psi}_{\nu_n}^2 \\ &= g^{\nu_1 \bar{\nu}_1} \dots g^{\nu_n \bar{\nu}_n} \left[ \nabla_{\bar{\mu}} F_{\mu_1 \dots \mu_m, \bar{\nu}_1 \dots \bar{\nu}_n} \right] \psi_1^{\mu_1} \dots \psi_1^{\mu_m} \bar{\psi}_{\nu_1}^2 \dots \bar{\psi}_{\nu_n}^2 \\ &= \nabla_{\bar{\mu}} F_{\mu_1 \dots \mu_m, \bar{\nu}_1 \dots \bar{\nu}_n} dx^{\mu_1} \dots dx^{\mu_m} d\bar{x}^{\bar{\nu}_1} \dots d\bar{x}^{\bar{\nu}_n}, \end{aligned} \quad (3.18)$$

where we used (3.16) and identified  $g^{\mu\bar{\nu}} \bar{\psi}_\mu^2 \sim d\bar{x}^{\bar{\nu}}$ . Let us notice that in the last two lines the covariant derivative  $\nabla_{\bar{\mu}}$  contains the needed connections  $\Gamma_{\bar{\nu}\bar{\sigma}}^{\bar{\lambda}}$  acting on the anti-holomorphic indices. We see then that the identification (3.17) is valid, keeping in mind the discussion above, also at the level of  $(m, n)$ -forms, and only when acting on  $(m, n)$ -forms the supercharges correctly reproduce the Dolbeault operators and their adjoints

$$\begin{aligned} iQ_1 &\sim \partial = dx^\mu \partial_\mu, & i\bar{Q}^2 &\sim \bar{\partial} = d\bar{x}^{\bar{\mu}} \bar{\partial}_{\bar{\mu}}, \\ i\bar{Q}^1 &\sim -\partial^\dagger = \bar{\nabla}^\mu \frac{\partial}{\partial(dx^\mu)}, & iQ_2 &\sim -\bar{\partial}^\dagger = \nabla^{\bar{\mu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\mu}})}, \end{aligned} \quad (3.19)$$

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<sup>2</sup>We denote with  $e$  the determinant of the vielbein  $e_\mu^a$ . See Appendix B in [5] for details.

with  $\bar{\nabla}^\mu = g^{\mu\bar{\nu}} \nabla_{\bar{\nu}}$  and  $\nabla^{\bar{\mu}} = g^{\nu\bar{\mu}} \nabla_\nu$ . The quantum  $U(d)$  generators (3.7) act as

$$M^{\mu\bar{\nu}} \sim g^{\nu\bar{\nu}} dx^\mu \frac{\partial}{\partial(dx^\nu)} - g^{\mu\bar{\mu}} d\bar{x}^{\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\mu}})} , \quad (3.20)$$

while the complete hamiltonian corresponds to the Laplace-Beltrami operator

$$H \sim -\Delta = \partial\partial^\dagger + \partial^\dagger\partial = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} \quad (3.21)$$

$$= -\frac{\nabla^2}{2} - \frac{1}{2} R_{\mu\bar{\nu}\lambda\bar{\sigma}} M^{\mu\bar{\nu}} M^{\lambda\bar{\sigma}} , \quad (3.22)$$

where  $\nabla^2$  is the  $2d$ -dimensional curved space laplacian

$$\nabla^2 = G^{MN} \nabla_M \nabla_N = g^{\mu\bar{\nu}} (\nabla_\mu \nabla_{\bar{\nu}} + \nabla_{\bar{\nu}} \nabla_\mu) . \quad (3.23)$$

Finally, the action of the off-diagonal  $R$ -symmetry generators (2.14) now corresponds to the insertion of the Kähler form (up to an imaginary factor) and to a trace with respect to the Kähler metric, respectively,

$$J_1^2 \sim g \wedge = g_{\mu\bar{\nu}} dx^\mu d\bar{x}^{\bar{\nu}} , \quad J_2^1 \sim -\text{Tr} = -g^{\mu\bar{\nu}} \frac{\partial^2}{\partial(dx^\mu)\partial(d\bar{x}^{\bar{\nu}})} . \quad (3.24)$$

We stress here that, since the holomorphic volume form  $e \epsilon_{\mu_1 \dots \mu_d}$  is not globally defined, unless the manifold is Calabi-Yau, the equivalence between the  $J_1^2$  and  $J_2^1$  constraints is affected by topological mismatches that appear when treating the  $\alpha$  forms, as already anticipated in Sec. 2.

Since the extended supersymmetry algebra has the same form as in flat space, the field equations of the four models keep the same form (2.26), (2.35), (2.38), and (2.39) in terms of the operators  $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger, \Delta, \text{Tr}$ . The difference consists in replacing  $\delta_{\mu\bar{\nu}}$  with  $g_{\mu\bar{\nu}}$  and their inverses, in the covariant  $(\partial^\dagger, \bar{\partial}^\dagger)$  operators given by (3.19), and in the full hamiltonian (3.21), realized as the full Laplace-Beltrami operator that also contains curvature corrections:  $\Delta = \frac{\nabla^2}{2} + \frac{1}{2} R_{\mu\bar{\nu}\lambda\bar{\sigma}} M^{\mu\bar{\nu}} M^{\lambda\bar{\sigma}}$ .

## 4 Effective action of differential forms

In the present section we wish to study the functional quantization of our spinning particle models coupled to a curved Kähler background. This will provide the heat kernel expansion of the effective actions for the different field theories of  $(p, q)$ -forms, as well as exact relations between Hodge dual descriptions of the same quantum theory. The proofs of exact dualities and the related topological issues will be carefully studied in the next section.

The effective action for such differential forms on Kähler spaces is recovered in the worldline formalism by quantizing the corresponding spinning particles on a circle (a one-dimensional torus). The worldline actions in phase space are obtained by coupling the correct subset of Noether charges  $\mathcal{C}_I^A$ , listed in the previous section for the different models, to worldline gauge fields, and are given by (3.5). By eliminating momenta in (3.5) and

performing a Wick rotation, we recover the euclidean actions in configuration space for the four theories, that explicitly read

$$\begin{aligned}
S_A &= \int_0^1 d\tau \left[ e^{-1} g_{\mu\bar{\nu}} (\dot{x}^\mu - \bar{\chi}^i \psi_i^\mu) (\dot{\bar{x}}^{\bar{\nu}} - \chi_i \bar{\psi}^{i\bar{\nu}}) + \bar{\psi}_\mu^1 [D_\tau + ia_1^1] \psi_1^\mu + is_1 a_1^1 \right. \\
&\quad \left. + \bar{\psi}_\mu^2 [D_\tau + ia_2^2] \psi_2^\mu + is_2 a_2^2 - \frac{e}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] \\
S_\alpha &= \int_0^1 d\tau \left[ e^{-1} g_{\mu\bar{\nu}} (\dot{x}^\mu - \bar{\chi}^i \psi_i^\mu) (\dot{\bar{x}}^{\bar{\nu}} - \chi_i \bar{\psi}^{i\bar{\nu}}) + \bar{\psi}_\mu^i [\delta_i^j D_\tau + ia_i^j] \psi_j^\mu + is a_i^i \right. \\
&\quad \left. - \frac{e}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] \\
S_B &= \int_0^1 d\tau \left[ e^{-1} g_{\mu\bar{\nu}} \dot{x}^\mu \dot{\bar{x}}^{\bar{\nu}} + \bar{\psi}_\mu^1 [D_\tau + ia_1^1] \psi_1^\mu + is_1 a_1^1 + \bar{\psi}_\mu^2 [D_\tau + ia_2^2] \psi_2^\mu + is_2 a_2^2 \right. \\
&\quad \left. - \frac{e}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] \\
S_\beta &= \int_0^1 d\tau \left[ e^{-1} g_{\mu\bar{\nu}} \dot{x}^\mu \dot{\bar{x}}^{\bar{\nu}} + \bar{\psi}_\mu^i [\delta_i^j D_\tau + ia_i^j] \psi_j^\mu + is a_i^i - \frac{e}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right],
\end{aligned} \tag{4.1}$$

where the covariant time derivative is given by  $D_\tau \psi_i^\mu = \dot{\psi}_i^\mu + \dot{x}^\nu \Gamma_{\nu\lambda}^\mu \psi_i^\lambda$ . Note that along with the Wick rotation  $t \rightarrow -i\tau$  we have also rotated the gauge fields  $a_j^i \rightarrow -ia_j^i$  to keep the gauge group compact.

The quantization of the spinning particle on a circle parameterized by the proper time  $\tau \in [0, 1]$ , gives the effective action for the differential forms we are interested in, coupled to the background metric  $g_{\mu\bar{\nu}}(x, \bar{x})$

$$Z_{\mathcal{A}}[g] \propto \int \left[ \frac{\mathcal{D}X \mathcal{D}G}{\text{Vol}(\text{Gauge})} \right]_{\mathcal{A}} e^{-S_{\mathcal{A}}[X, G_{\mathcal{A}}]}. \tag{4.2}$$

We denote the dynamical variables as  $X = (x^\mu, \bar{x}^{\bar{\mu}}, \psi_i^\mu, \bar{\psi}_\mu^i)$ , while  $G_{\mathcal{A}}$  is the subset of  $(e, \bar{\chi}^i, \chi_i, a_j^i)$  needed for the model  $\mathcal{A}$ . The subscript  $\mathcal{A}$  in the functional measure stands for the dependence of the measure itself on the choice of the model, *i.e.*, on the choice of the worldline gauge group.

By means of the Faddeev-Popov procedure, we gauge fix the “supergravity multiplet” to the constant values

$$\tilde{G} = \left( \beta, 0, 0, \begin{pmatrix} \phi & 0 \\ 0 & \theta \end{pmatrix} \right),$$

and we are left with modular integrations over  $\beta, \phi$  and  $\theta$ , with the one-loop measure that was carefully studied in [3, 26]

$$Z_{\mathcal{A}}[g] \propto \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \tilde{\mu}_{\mathcal{A}}(\phi, \theta) \int_{\text{P}} \mathcal{D}x \mathcal{D}\bar{x} \int_{\text{A}} D\bar{\psi} D\psi e^{-S_{\mathcal{A}}[X, \tilde{G}_{\mathcal{A}}]} \tag{4.3}$$

where  $S[X, \tilde{G}_{\mathcal{A}}]$  stands for the gauge fixed action, *i.e.* (4.1) evaluated at  $G = \tilde{G}$ . The subscript P and A denote periodic and antiperiodic boundary conditions, respectively. The integral over  $\beta$  is the usual proper time integral with the well known one-loop measure. The term  $\tilde{\mu}_{\mathcal{A}}(\phi, \theta)$  contains the Faddeev-Popov factors, depending on the chosen model:

when the supersymmetry is gauged ( $A$  and  $\alpha$  models), the bosonic superghosts associated to  $\chi_i$  and  $\bar{\chi}^i$  produce a factor  $\left(2 \cos \frac{\phi}{2} 2 \cos \frac{\theta}{2}\right)^{-2}$ , and when the whole  $U(2)$   $R$ -symmetry is gauged ( $\alpha$  and  $\beta$  models), the ghosts of the non-abelian group produce the additional term  $\frac{1}{2} \left(2 \sin \frac{\phi-\theta}{2}\right)^2$ , see *e.g.* [3, 26].

We denote with  $\mathcal{D}x$  the general coordinate invariant measure, *i.e.*

$$\mathcal{D}x \mathcal{D}\bar{x} \sim \prod_{\tau=0}^1 d^d x(\tau) d^d \bar{x}(\tau) g(x(\tau), \bar{x}(\tau)) ,$$

with  $g = \det g_{\mu\bar{\nu}}$ , while  $D\psi \sim \prod_{\tau=0}^1 d^d \psi(\tau)$  is the translational invariant flat measure.<sup>3</sup> Formula (4.3) gives the worldline representation of the effective action for the differential form  $\mathcal{A}$ . At this stage, all the dependence on the model chosen is contained in the modular measure  $\tilde{\mu}_{\mathcal{A}}(\phi, \theta)$  and in the Chern-Simons part of the action. To exploit it further, we need some more manipulations. In order to extract the integral over spacetime, we choose an arbitrary  $x_0$  as a base-point for our loops. The periodic path integral then factorizes as  $\int_{\mathbb{P}} \mathcal{D}x \mathcal{D}\bar{x} = \int d^d x_0 d^d \bar{x}_0 g(x_0) \int_{x(0)=x(1)=x_0} \mathcal{D}x \mathcal{D}\bar{x}$ , and we let the coordinate fields fluctuate around the fixed point  $x_0$  as:  $x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$ , with  $q^\mu(0) = q^\mu(1) = 0$ . The  $x$  path integral becomes  $\int_{\mathbb{D}} \mathcal{D}q \mathcal{D}\bar{q}$ , where  $\mathbb{D}$  stands for Dirichlet boundary conditions. The remaining obstacle in performing a perturbative expansion is the field dependent measure  $\mathcal{D}q \mathcal{D}\bar{q}$ . Following [27, 28] we exponentiate the metric determinant with a path integral over fermionic complex ghosts  $b^\mu$  and  $\bar{c}^{\bar{\nu}}$ :  $\mathcal{D}q \mathcal{D}\bar{q} = Dq D\bar{q} \int Db D\bar{c} e^{-S_{\text{gh}}}$ . The full gauge fixed action, containing the ghost term:  $S_{\mathcal{A}}^{\text{gf}} \equiv S_{\mathcal{A}}[X, \tilde{G}_{\mathcal{A}}] + S_{\text{gh}}$  takes the following form<sup>4</sup>

$$\begin{aligned} S_{\mathcal{A}}^{\text{gf}} = & \frac{1}{\beta} \int_0^1 d\tau \left[ g_{\mu\bar{\nu}} (\dot{q}^\mu \dot{\bar{q}}^{\bar{\nu}} + b^\mu \bar{c}^{\bar{\nu}}) + \bar{\psi}_\mu^1 (D_\tau + i\phi) \psi_1^\mu + \bar{\psi}_\mu^2 (D_\tau + i\theta) \psi_2^\mu \right. \\ & \left. - \frac{1}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] + \tilde{S}_{\mathcal{A}}^{CS} , \end{aligned} \quad (4.4)$$

where  $\tilde{S}_{\mathcal{A}}^{CS}$  is the gauge fixed euclidean Chern-Simons term, that we choose to plug in the measure  $\tilde{\mu}_{\mathcal{A}}$ .

In order to perform perturbative calculations we expand all background fields around the fixed point  $x_0$ . The action written above splits into a quadratic part  $S_2$  giving propagators

$$S_2 = \frac{1}{\beta} \int_0^1 d\tau \left[ g_{\mu\bar{\nu}}(x_0) (\dot{q}^\mu \dot{\bar{q}}^{\bar{\nu}} + b^\mu \bar{c}^{\bar{\nu}}) + \bar{\psi}_\mu^1 (\partial_\tau + i\phi) \psi_1^\mu + \bar{\psi}_\mu^2 (\partial_\tau + i\theta) \psi_2^\mu \right] , \quad (4.5)$$

and an interaction part  $S_{\text{int}}$

$$\begin{aligned} S_{\text{int}} = & \frac{1}{\beta} \int_0^1 d\tau \left[ (g_{\mu\bar{\nu}}(x_0 + q) - g_{\mu\bar{\nu}}(x_0)) (\dot{q}^\mu \dot{\bar{q}}^{\bar{\nu}} + b^\mu \bar{c}^{\bar{\nu}}) + \dot{q}^\nu \Gamma_{\nu\lambda}^\mu(x_0 + q) \bar{\psi}_\mu^i \psi_i^\lambda \right. \\ & \left. - \frac{1}{2} R_\mu{}^\nu{}_\lambda{}^\sigma(x_0 + q) \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] . \end{aligned} \quad (4.6)$$

<sup>3</sup>Note that, since  $\psi$ 's are spacetime vectors, while  $\bar{\psi}$ 's are covectors, the covariant measure coincides with the flat one:  $\mathcal{D}\bar{\psi} \mathcal{D}\psi = D\bar{\psi} D\psi$ .

<sup>4</sup>We rescaled fermions by  $\psi \rightarrow \frac{1}{\sqrt{\beta}} \psi$  to have common normalizations of the two-point functions.



We denote as  $\langle \bullet \rangle$  the quantum average over the quadratic action

$$\langle \bullet \rangle = \frac{\int Dq D\bar{q} Db D\bar{c} D\bar{\psi} D\psi \bullet e^{-S_2}}{\int Dq D\bar{q} Db D\bar{c} D\bar{\psi} D\psi e^{-S_2}}.$$

The partition function (4.3) finally reads

$$Z_A[g] \propto \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \mu_A(\phi, \theta) \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \langle e^{-S_{\text{int}}} \rangle, \quad (4.7)$$

where  $(2\pi\beta)^{-d}$  is the usual free bosonic path integral, and we have plugged the Chern-Simons term as well as the free fermionic path integral in the modular measure

$$\mu_A(\phi, \theta) = \tilde{\mu}_A(\phi, \theta) e^{-\tilde{S}_A^{CS}} \left(2 \cos \frac{\phi}{2}\right)^d \left(2 \cos \frac{\theta}{2}\right)^d. \quad (4.8)$$

Looking at (4.7), we see that the expression we obtained is quite compact: indeed, the perturbative computation of  $\langle e^{-S_{\text{int}}} \rangle$  is common to all the four models we are interested in, and the choice of the model is entirely encoded in the form of the modular factor  $\mu_A(\phi, \theta)$ , that is time to make more explicit. Let us consider the  $A$  theory: it gauges the supersymmetries and only the  $U(1) \times U(1)$  subgroup. Since the  $(p, q)$ -form comes from a  $(p+1, q+1)$  field strength,  $F_{(p+1, q+1)} = \partial \bar{\partial} A_{(p, q)}$ , according to (2.29) the Chern-Simons couplings are given by  $s_1 = p+1-d/2$  and  $s_2 = d/2-q-1$ , and the corresponding measure reads

$$\mu_A(\phi, \theta) = e^{-i(p+1-d/2)\phi} e^{i(q+1-d/2)\theta} \left(2 \cos \frac{\phi}{2}\right)^{d-2} \left(2 \cos \frac{\theta}{2}\right)^{d-2}. \quad (4.9)$$

On the other hand, the  $\alpha$  model has a single Chern-Simons coupling (see (2.20)), but gauges the whole  $U(2)$  group, giving the measure

$$\mu_\alpha(\phi, \theta) = \frac{1}{2} e^{-i(p+1-d/2)(\phi+\theta)} \left(2 \cos \frac{\phi}{2}\right)^{d-2} \left(2 \cos \frac{\theta}{2}\right)^{d-2} \left(2 \sin \frac{\phi-\theta}{2}\right)^2. \quad (4.10)$$

In the remaining cases, where the supersymmetries are not gauged, the forms do not come from a field strength, and one has shifted Chern-Simons couplings:  $s_1 = p-d/2$ ,  $s_2 = d/2-q$ . The resulting modular factors have the following form:

$$\mu_B(\phi, \theta) = e^{-i(p-d/2)\phi} e^{i(q-d/2)\theta} \left(2 \cos \frac{\phi}{2}\right)^d \left(2 \cos \frac{\theta}{2}\right)^d, \quad (4.11)$$

$$\mu_\beta(\phi, \theta) = \frac{1}{2} e^{-i(p-d/2)(\phi+\theta)} \left(2 \cos \frac{\phi}{2}\right)^d \left(2 \cos \frac{\theta}{2}\right)^d \left(2 \sin \frac{\phi-\theta}{2}\right)^2. \quad (4.12)$$

We are now ready to compute  $\langle e^{-S_{\text{int}}} \rangle$  up to order  $\beta^2$ . Since we are free to use any coordinate system, we choose to employ Kähler normal coordinates (see [29], for instance) centered at  $x_0$ , that allow to maintain explicit covariance under reparametrization of  $x_0$  while keeping holomorphic coordinates at each step. Denoting with  $S_n$  the part of  $S_{\text{int}}$  containing  $n$ -fields vertices, it turns out that the only terms giving non vanishing contribution

up to order  $\beta^2$  are the following ones

$$\begin{aligned}
S_4 &= \frac{1}{\beta} \int_0^1 d\tau \left[ R_{\mu\bar{\nu}\lambda\bar{\sigma}} q^\lambda \bar{q}^{\bar{\sigma}} (\dot{q}^\mu \bar{\dot{q}}^{\bar{\nu}} + b^\mu \bar{c}^{\bar{\nu}}) - R^\lambda_{\sigma\bar{\nu}\mu} \dot{q}^\mu \bar{q}^{\bar{\nu}} \psi_i^\sigma \bar{\psi}_\lambda^i - \frac{1}{2} R_\mu{}^\nu{}_\lambda{}^\sigma \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right] \\
S_6 &= \frac{1}{\beta} \int_0^1 d\tau \left[ \frac{1}{4} [\nabla_{(\bar{\sigma}} \nabla_\lambda R_{\mu\bar{\nu}\rho\bar{\kappa})} + 3R^\bar{\tau}_{(\bar{\nu}\lambda\bar{\kappa}} R_{\mu\bar{\sigma}\rho)}] q^\lambda \bar{q}^{\bar{\sigma}} q^\rho \bar{q}^{\bar{\kappa}} (\dot{q}^\mu \bar{\dot{q}}^{\bar{\nu}} + b^\mu \bar{c}^{\bar{\nu}}) \right. \\
&\quad \left. - \frac{1}{2} [\nabla_\rho \nabla_{\bar{\sigma}} R^\lambda_{\mu\bar{\lambda}\nu} + R^\bar{\tau}_{\bar{\lambda}\rho\bar{\sigma}} R^\lambda_{\mu\bar{\tau}\nu}] q^\rho \bar{q}^{\bar{\sigma}} \bar{q}^{\bar{\lambda}} \dot{q}^\mu \psi_i^\nu \bar{\psi}_\lambda^i - \frac{1}{2} \nabla_\rho \nabla_{\bar{\tau}} R_\mu{}^\nu{}_\lambda{}^\sigma q^\rho \bar{q}^{\bar{\tau}} \psi_i^\mu \bar{\psi}_\nu^i \psi_j^\lambda \bar{\psi}_\sigma^j \right], \tag{4.13}
\end{aligned}$$

where all tensors are calculated at  $x_0$  and round brackets denote weighted symmetrization, separately among holomorphic and anti-holomorphic indices.

The two-point functions are readily computed from the free action, and are given by

$$\begin{aligned}
\langle q^\mu(\tau) \bar{q}^{\bar{\nu}}(\sigma) \rangle &= -\beta g^{\mu\bar{\nu}}(x_0) \Delta(\tau, \sigma), \quad \langle b^\mu(\tau) \bar{c}^{\bar{\nu}}(\sigma) \rangle = -\beta g^{\mu\bar{\nu}}(x_0) \delta(\tau, \sigma) \\
\langle \psi_i^\mu(\tau) \bar{\psi}_\nu^j(\sigma) \rangle &= \beta \delta_i^j \delta_\nu^\mu \Delta_f(\tau - \sigma, \phi_i)
\end{aligned} \tag{4.14}$$

where  $\phi_i \equiv (\phi, \theta)$ , and the propagators in the continuum limit read

$$\begin{aligned}
\Delta(\tau, \sigma) &= \sigma(\tau - 1) \theta(\tau - \sigma) + \tau(\sigma - 1) \theta(\sigma - \tau), \\
\Delta_f(\tau - \sigma, \phi_i) &= \frac{e^{-i\phi_i(\tau - \sigma)}}{2 \cos \frac{\phi_i}{2}} [e^{i\frac{\phi_i}{2}} \theta(\tau - \sigma) - e^{-i\frac{\phi_i}{2}} \theta(\sigma - \tau)]
\end{aligned} \tag{4.15}$$

with  $\theta(\tau - \sigma)$  the Heaviside step function and  $\delta(\tau, \sigma)$  the Dirac delta acting on functions that vanish at the endpoints. It is well known that path integrals in curved space require regularization. Indeed one can see from (4.15) that in the computations one has to face products and derivatives of such distributions, that generically are ill defined. Here we choose to employ Time Slicing (TS) regularization [30–32], that gives unambiguous prescriptions on how to handle these subtleties, and does not require counterterms (the standard TS counterterm of the  $N = 2$  sigma model vanishes on Kähler manifolds). Among the usual regularization schemes the TS rules are the simpler<sup>5</sup>: when computing the various Feynman diagrams the delta functions have to be treated as Kronecker deltas, and the Heaviside theta has the regulated value  $\theta(0) = \frac{1}{2}$ . Moreover, the ghost system forbids the appearance of products of delta functions. By means of these prescriptions, the propagators (4.15) and their derivatives have well defined equal-time expressions, that are listed for convenience in Appendix B

From (4.13)–(4.14) we see that each piece  $S_n$  of  $S_{\text{int}}$  gives a contribution of order  $\beta^{n/2-1}$ . Therefore, our quantum average can be written explicitly as

$$\langle e^{-S_{\text{int}}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2} \langle S_4^2 \rangle + \mathcal{O}(\beta^3). \tag{4.16}$$

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<sup>5</sup> Other known regularizations are Mode Regularization (MR) [33–35] and Dimensional Regularization (DR) [36–38]. MR carries a non covariant counterterm that is non vanishing on Kähler manifolds, so that additional vertices must be included to obtain the correct final answer. DR is covariant, but requires a few integration by parts for its implementation, so that we found TS to be the simplest one for the present calculations. All of these regularizations have been recently extended to nonlinear sigma models with  $N$  supersymmetries in [39].

Using the expressions given in (4.13) and TS prescriptions in calculating Feynman diagrams, one eventually obtains

$$\begin{aligned}
\langle e^{-S_{\text{int}}} \rangle = & 1 + \beta \left( -\frac{1}{12} + \frac{1}{4} \tan \frac{\phi}{2} \tan \frac{\theta}{2} \right) R \\
& + \beta^2 \left\{ \left[ \frac{1}{180} - \frac{1}{96} \left( \cos^{-2} \frac{\phi}{2} + \cos^{-2} \frac{\theta}{2} \right) + \frac{1}{32} \cos^{-2} \frac{\phi}{2} \cos^{-2} \frac{\theta}{2} \right] R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} \right. \\
& + \left[ -\frac{17}{720} + \frac{1}{24} \left( \cos^{-2} \frac{\phi}{2} + \cos^{-2} \frac{\theta}{2} \right) - \frac{1}{16} \cos^{-2} \frac{\phi}{2} \cos^{-2} \frac{\theta}{2} + \frac{1}{48} \tan \frac{\phi}{2} \tan \frac{\theta}{2} \right] R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} \\
& + \left[ \frac{5}{144} - \frac{1}{32} \left( \cos^{-2} \frac{\phi}{2} + \cos^{-2} \frac{\theta}{2} \right) + \frac{1}{32} \cos^{-2} \frac{\phi}{2} \cos^{-2} \frac{\theta}{2} - \frac{1}{48} \tan \frac{\phi}{2} \tan \frac{\theta}{2} \right] R^2 \\
& \left. + \left[ -\frac{1}{240} + \frac{1}{48} \tan \frac{\phi}{2} \tan \frac{\theta}{2} \right] \nabla^2 R \right\}, \tag{4.17}
\end{aligned}$$

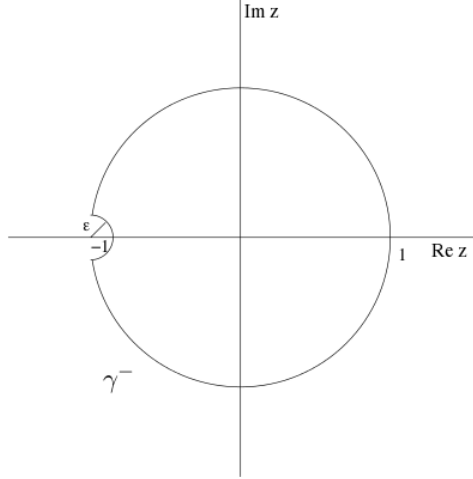
where  $\nabla^2 R = 2g^{\mu\bar{\nu}} \partial_\mu \partial_{\bar{\nu}} R$ . This is our perturbative master formula, from which we can easily compute the first Seeley-DeWitt coefficients (SDW or heat kernel coefficients) for our different models by plugging (4.17) into (4.3), and performing the modular integrals over  $\phi$  and  $\theta$  with the different measures  $\mu_{\mathcal{A}}(\phi, \theta)$ . At this stage a first subtlety arises in performing such integrations: by looking at the measures  $\mu_{\mathcal{A}}$  and at the expansion (4.17), one sees that, given the complex dimension  $d$ , at sufficiently large orders in  $\beta$  two poles appear along the integration paths, namely at  $\phi = \pi$  and  $\theta = \pi$ . This corresponds to effectively giving periodic boundary conditions to fermionic fields that develop zero modes one has to deal with (see *e.g.* [6]). In order to find and understand the correct prescription, it is useful to switch to Wilson loop variables, *i.e.*  $z = e^{i\phi}$  and  $w = e^{i\theta}$ . The modular integrals then turn into contour integrals on the unit circle centered in zero in the complex  $z$  and  $w$  planes, that we denote by  $\gamma$

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} = \oint_{\gamma} \frac{dz}{2\pi iz} \oint_{\gamma} \frac{dw}{2\pi iw}.$$

The possible poles now show up on the integration contour at  $z = -1$  or  $w = -1$ . To correctly deal with such poles, it turns out that one has to slightly deform the two contours excluding the pole in  $z = -1$ , with the regulated contour  $\gamma^-$  shown in figure 1, and including the pole in  $w = -1$  with the regulated contour  $\gamma^+$  shown in figure 2. The heuristic reason for this choice is that the correct known results, for instance the SDW coefficients for a scalar field, only come out from the aforementioned prescription. Nevertheless, we can justify the choice of two different contours for the  $z$  and  $w$  integrals: it corresponds to using two different bases for the two fermionic species. While the first one is realized treating  $\psi_1$ 's as creation and  $\bar{\psi}^1$ 's as annihilation operators, the second has  $\bar{\psi}^2$ 's as creators and  $\psi_2$ 's as annihilators. The first realization leads to the “standard” contour  $\gamma^-$  for  $z$  (see [5]), while the second, that is obtained from the usual one with an anti-holomorphic dualization of the fermionic vacuum<sup>6</sup>, leads to the  $\gamma^+$  contour for  $w$ . Hence, at the level of differential

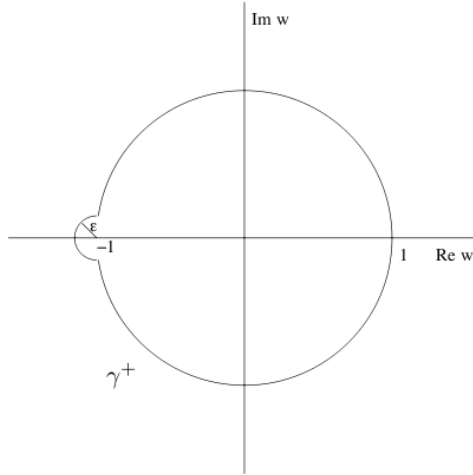
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<sup>6</sup>Denoting with  $|0\rangle_{\bar{\psi}^2}$  the Fock vacuum annihilated by the  $\bar{\psi}^2$  operators, and with  $|0\rangle_{\psi_2}$  the Fock vacuum annihilated by the  $\psi_2$ 's, one has  $|0\rangle_{\psi_2} \propto e^{\epsilon_{\mu_1 \dots \mu_d} \psi_2^{\mu_1} \dots \psi_2^{\mu_d}} |0\rangle_{\bar{\psi}^2}$ . Note that in the main text we use bra coherent states, dual to ket coherent states built from  $|0\rangle_{\psi_2}$ , so that throughout the paper the  $\bar{\psi}^2$ 's indicate the eigenvalues (Grassmann numbers) of the corresponding operators, and no confusion should arise.



**Figure 1.** The regulated contour  $\gamma^-$  that excludes the pole at  $z = -1$ .

forms, switching from  $\gamma^-$  to  $\gamma^+$  contour corresponds to performing an (anti)-holomorphic Hodge duality<sup>7</sup>. As shown in [5], this introduces an extra coupling of the fields to the  $U(1)$  part of the Kähler connection, that is not allowed in the present models.



**Figure 2.** The regulated contour  $\gamma^+$  that includes the pole at  $w = -1$ .

Given the correct contour prescriptions, we are now ready to compute the SDW coefficients for the differential forms and study their duality properties. We shall define the SDW coefficients in the expansion of the partition functions as:

$$Z \propto \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \left\{ v_1 + v_2 \beta R + \beta^2 \left[ v_3 R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} + v_4 R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + v_5 R^2 + v_6 \nabla^2 R \right] \right\}. \quad (4.18)$$

<sup>7</sup>The total Hodge duality involves the full  $\epsilon$  tensor  $g \epsilon_{\mu_1 \dots \mu_d \bar{\nu}_1 \dots \bar{\nu}_d}$ , while the holomorphic one requires only  $e \epsilon_{\mu_1 \dots \mu_d}$

Let us notice that in standard quantum field theories the first coefficient  $v_1$  gives the number of physical degrees of freedom, and vanishes when one treats non-propagating fields. A similar interpretation is applicable in our case, where a Kähler manifold takes the role of spacetime. We will list the coefficients for a generic form  $\mathcal{A}$  as:  $\mathcal{A} \rightarrow (v_1; v_2; v_3; v_4; v_5; v_6)$

Let us consider first the  $A$  model of  $(p, q)$ -form gauge fields. Plugging the master formula (4.17) in (4.3) with the measure (4.9) we have, using the Wilson loop variables,

$$Z_A[g] \propto \int_0^\infty \frac{d\beta}{\beta} \oint_{\gamma^-} \frac{dz}{2\pi i z} \oint_{\gamma^+} \frac{dw}{2\pi i w} \frac{1}{z^p w^{d-2-q}} [(z+1)(w+1)]^{d-2} \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \langle e^{-S_{\text{int}}} \rangle. \quad (4.19)$$

Upon performing the modular integrations one gets the following SDW coefficients

$$\begin{aligned} A_{(p,q)} \rightarrow & \binom{d-2}{p} \binom{d-2}{q} \times \left( 1; \frac{1}{6} - \frac{p(d-2-q) + q(d-2-p)}{2(d-2)^2}; \right. \\ & \frac{1}{180} - \frac{p(d-2-p) + q(d-2-q)}{24(d-2)(d-3)} + \frac{p(d-2-p)q(d-2-q)}{2(d-2)^2(d-3)^2}; \\ & -\frac{1}{360} + \frac{p(d-2-p) + q(d-2-q)}{6(d-2)(d-3)} - \frac{p(d-2-p)q(d-2-q)}{(d-2)^2(d-3)^2} - \frac{p(d-2-q) + q(d-2-p)}{24(d-2)^2}; \\ & \frac{1}{72} - \frac{p(d-2-p) + q(d-2-q)}{8(d-2)(d-3)} + \frac{p(d-2-p)q(d-2-q)}{2(d-2)^2(d-3)^2} + \frac{p(d-2-q) + q(d-2-p)}{24(d-2)^2}; \\ & \left. \frac{1}{60} - \frac{p(d-2-q) + q(d-2-p)}{24(d-2)^2} \right). \end{aligned} \quad (4.20)$$

This formula is valid for  $0 \leq p, q \leq d-2$ , since for larger  $p$  or  $q$  the field does not propagate, and for  $d \geq 4$ , since in lower dimensions the pole in  $z, w = -1$  appears also in these first coefficients, and one has to use the regulated contours. We can see that (4.20) correctly reproduces the coefficients for a scalar field, obtained by setting  $p = q = 0$

$$A_{(0,0)} \rightarrow \left( 1; \frac{1}{6}; \frac{1}{180}; -\frac{1}{360}; \frac{1}{72}; \frac{1}{60} \right), \quad (4.21)$$

as well as for  $(p, 0)$ -forms, as we can compare (4.20) for  $A_{(p,0)}$  with the results of [5] with the charge set to zero<sup>8</sup>

$$\begin{aligned} A_{(p,0)} \rightarrow & \binom{d-2}{p} \times \left( 1; \frac{1}{6} - \frac{p}{2(d-2)}; \frac{1}{180} - \frac{p(d-2-p)}{24(d-2)(d-3)}; \right. \\ & -\frac{1}{360} + \frac{p(d-2-p)}{6(d-2)(d-3)} - \frac{p}{24(d-2)}; \frac{1}{72} - \frac{p(d-2-p)}{8(d-2)(d-3)} + \frac{p}{24(d-2)}; \\ & \left. \frac{1}{60} - \frac{p}{24(d-2)} \right). \end{aligned} \quad (4.22)$$

---

<sup>8</sup>For  $q = 0$  or  $p = 0$  the compensator term drops out from the field equation of our  $A_{(p,q)}$  forms, that reduce to the  $(p, 0)$ -forms studied in [5] with zero charge.

By inspecting the coefficients (4.20), it is manifest the symmetry under the exchange  $p \leftrightarrow q$  that corresponds to complex conjugation:  $A_{(p,q)} \sim A_{(q,p)}$ , and under  $p \leftrightarrow d-2-q$ , that corresponds to Hodge duality  $A_{(p,q)} \sim A_{(d-2-q, d-2-p)}$ . We will see in the following that the first symmetry is fully preserved in the effective action, while the Hodge duality suffers a topological mismatch that will be investigated at the non-perturbative level in the next section.

We now turn to computing the SDW coefficients in lower dimensions. In three complex dimensions propagating fields have  $p$  and  $q$  restricted to be zero or one, and we get

$$d = 3, \quad p, q = \{0, 1\}$$

$$A_{(p,q)} \rightarrow \left( 1; \frac{1}{6} - \frac{p(1-q) + q(1-p)}{2}; \frac{1}{180} - \frac{p+q}{24} + \frac{1}{2}pq; \right. \\ \left. - \frac{1}{360} + \frac{p+q}{6} - pq - \frac{p(1-q) + q(1-p)}{24}; \right. \\ \left. \frac{1}{72} - \frac{p+q}{8} + \frac{1}{2}pq + \frac{p(1-q) + q(1-p)}{24}; \frac{1}{60} - \frac{p(1-q) + q(1-p)}{24} \right). \quad (4.23)$$

It can be successfully compared with the results of [5] for the scalar and the  $(1,0)$  form setting the charge appearing there to zero. The formula (4.23) is still invariant under the exchange of  $p$  and  $q$ , that tells  $A_{(1,0)} \sim A_{(0,1)}$ , but the symmetry under Hodge duality  $p \rightarrow 1-q$ ,  $q \rightarrow 1-p$  is lost, due to topological mismatches, and indeed one has

$$d = 3$$

$$A_{(0,0)} \rightarrow \left( 1; \frac{1}{6}; \frac{1}{180}; -\frac{1}{360}; \frac{1}{72}; \frac{1}{60} \right), \quad (4.24)$$

$$A_{(1,1)} \rightarrow \left( 1; \frac{1}{6}; \frac{19}{45}; -\frac{241}{360}; \frac{19}{72}; \frac{1}{60} \right).$$

In  $d = 2$  only the scalar propagates, and by means of (4.19) one finds the correct results already presented

$$d = 2, \quad A_{(0,0)} \rightarrow \left( 1; \frac{1}{6}; \frac{1}{180}; -\frac{1}{360}; \frac{1}{72}; \frac{1}{60} \right). \quad (4.25)$$

We are not interested in  $d = 1$  in the present paper, since the field equations are somewhat degenerate from the very beginning, see *e.g.* [5].

Now we can turn to the  $\alpha$  model that describes  $(p, d-2-p)$ -form gauge fields. It is worth stressing that the present theory is definitely not obtainable by setting  $q = d-2-p$  in the  $A_{(p,q)}$  forms, since they obey different field equations. Indeed, by means of the measure (4.10), the partition function is given by

$$Z_\alpha[g] \propto -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \oint_{\gamma^-} \frac{dz}{2\pi i z} \oint_{\gamma^+} \frac{dw}{2\pi i w} \frac{1}{(zw)^{p+1}} [(z+1)(w+1)]^{d-2} (z-w)^2 \\ \times \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \langle e^{-S_{\text{int}}} \rangle. \quad (4.26)$$

The corresponding Seeley-DeWitt coefficients are given by

$$\begin{aligned}
\alpha_{(p,d-2-p)} &\rightarrow \binom{d-2}{p} \frac{(d-1)!}{(p+1)!(d-1-p)!} \times \left( 1; -\frac{1}{3} + \frac{p(d-2-p)}{(d-1)(d-2)}; \right. \\
&\frac{1}{180} + \frac{p(d-2-p)}{12(d-1)(d-2)^2(d-3)} [6(p+1)(d-1-p) - d(d-2)]; \\
&-\frac{2}{45} + \frac{p(d-2-p)}{12(d-1)(d-2)^2(d-3)} [(d-2)(5d-3) - 12(p+1)(d-1-p)]; \quad (4.27) \\
&\left. \frac{1}{18} + \frac{p(d-2-p)}{12(d-1)(d-2)^2(d-3)} [6(p+1)(d-1-p) - (d-2)(4d-3)]; \right. \\
&\left. -\frac{1}{40} + \frac{p(d-2-p)}{12(d-1)(d-2)} \right).
\end{aligned}$$

We notice that (4.27) is explicitly invariant under  $p \leftrightarrow d-2-p$ , corresponding to complex conjugation:  $\alpha_{(p,d-2-p)} \sim \alpha_{(d-2-p,p)}$ . In this model the Hodge duality acts trivially, since it sends  $\alpha_{(p,d-2-p)}$  into itself and indeed no topological mismatch can ever arise in this context, as we will show in more detail in the next section. As in the previous case this formula is valid in  $d \geq 4$  and for  $0 \leq p \leq d-2$ , and the lower dimensional cases will be presented separately. We notice that this model never describes a scalar field, except in two complex dimensions. The only check we can do is for  $p=0, d-2$ , since in this particular case the  $\alpha_{(d-2,0)}$  form should be the same as the  $A_{(d-2,0)}$ . In  $d \geq 4$  the check is indeed successful, and matches also the result of [5]:

$$d \geq 4, \quad \alpha_{(d-2,0)} \rightarrow \left( 1; -\frac{1}{3}; \frac{1}{180}; -\frac{2}{45}; \frac{1}{18}; -\frac{1}{40} \right), \quad (4.28)$$

however, this point contains a nontrivial subtlety. From Sec. 2 one can see that the difference between the field equations for  $\alpha$  and  $A$  forms is essentially in the trace term. Therefore, it is natural to think that, at the level of effective actions, one should be able to obtain the  $\alpha$  contribution from the  $A$  model by subtracting some trace terms. This is indeed the case, as we will prove in sec. 5.1, and one has the following exact relation

$$Z_{p,d-2-p}^\alpha = Z_{p,d-2-p}^A - \frac{1}{2} Z_{p-1,d-3-p}^A - \frac{1}{2} Z_{p+1,d-1-p}^A, \quad (4.29)$$

where we see that the subtractions involve the trace and its dual. The subtlety arises for  $p=0$  or  $p=d-2$ : in this case, one would naively expect the subtraction terms to be zero. While this is true for negative degree forms, the term  $Z_{1,d-1}^A$  is nonzero, although non-propagating and purely topological, and the effective action becomes:

$$Z_{0,d-2}^\alpha = Z_{0,d-2}^A - \frac{1}{2} Z_{1,d-1}^{A \text{ top}}, \quad (4.30)$$

where we stressed the topological nature of  $A_{(1,d-1)}$ .

This topological contribution is not visible in  $d \geq 4$  at order  $\beta^2$ , but one can find it in lower dimensions. For instance, in three complex dimensions the only physical  $\alpha$  field is  $\alpha_{(1,0)} \sim \alpha_{(0,1)}$ , with SDW coefficients given by

$$d = 3$$

$$\alpha_{(0,1)} \rightarrow \left( 1; -\frac{1}{3}; \frac{139}{720}; -\frac{53}{180}; \frac{17}{144}; -\frac{1}{40} \right), \quad (4.31)$$

that should naively equal

$$d = 3$$

$$A_{(0,1)} \rightarrow \left( 1; -\frac{1}{3}; -\frac{13}{360}; \frac{11}{90}; -\frac{5}{72}; -\frac{1}{40} \right), \quad (4.32)$$

given by (4.23). The mismatch is due to the topological  $A_{(1,2)}$  form, whose contribution can be computed from (4.19) and reads

$$d = 3$$

$$A_{(1,2)} \rightarrow \left( 0; 0; -\frac{11}{24}; \frac{5}{6}; -\frac{3}{8}; 0 \right), \quad (4.33)$$

indeed satisfying (4.30), that tells

$$Z_{0,1}^\alpha = Z_{0,1}^A - \frac{1}{2} Z_{1,2}^{A^{\text{top}}}. \quad (4.34)$$

The other check we can perform is in  $d = 2$ , where the only physical field should be a scalar  $\alpha_{(0,0)}$ . Its coefficients are

$$d = 2$$

$$\alpha_{(0,0)} \rightarrow \left( 1; -\frac{1}{3}; -\frac{11}{45}; \frac{41}{90}; -\frac{7}{36}; -\frac{1}{40} \right), \quad (4.35)$$

that differ from the usual scalar ones (4.21) by the topological contribution of  $A_{(1,1)}$

$$d = 2$$

$$A_{(1,1)} \rightarrow \left( 0; 1; \frac{1}{2}; -\frac{11}{12}; \frac{5}{12}; \frac{1}{12} \right), \quad (4.36)$$

and satisfy

$$Z_{0,0}^\alpha = Z_{0,0}^A - \frac{1}{2} Z_{1,1}^{A^{\text{top}}}. \quad (4.37)$$

We concentrate now on the remaining models, *i.e.* non gauge differential forms. In fact, the  $B$  and  $\beta$  forms do not enjoy any gauge symmetry, and obey simple wave equations given by the Laplace operator acting on forms. Let us start with the  $B$  model containing



non gauge  $(p, q)$ -forms  $B_{(p,q)}$ . The first measure of (4.11) allows us to write the partition function as:

$$Z_B[g] \propto \int_0^\infty \frac{d\beta}{\beta} \oint_\gamma \frac{dz}{2\pi iz} \oint_\gamma \frac{dw}{2\pi iw} \frac{1}{z^p w^{d-q}} [(z+1)(w+1)]^d \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \langle e^{-S_{\text{int}}} \rangle. \quad (4.38)$$

We denoted the two contours as  $\gamma$  since, as we will show in the following, the absence of the ghost terms for the supersymmetry avoids the presence of poles along the contour at the non-perturbative level. The effective action of these non gauge  $(p, q)$ -forms is characterized by the following coefficients, that we compute from (4.38)

$$\begin{aligned} B_{(p,q)} \rightarrow & \binom{d}{p} \binom{d}{q} \times \left( 1; \frac{1}{6} - \frac{p(d-q) + q(d-p)}{2d^2}; \right. \\ & \frac{1}{180} - \frac{p(d-p) + q(d-q)}{24d(d-1)} + \frac{p(d-p)q(d-q)}{2d^2(d-1)^2}; \\ & -\frac{1}{360} + \frac{p(d-p) + q(d-q)}{6d(d-1)} - \frac{p(d-p)q(d-q)}{d^2(d-1)^2} - \frac{p(d-q) + q(d-p)}{24d^2}; \\ & \frac{1}{72} - \frac{p(d-p) + q(d-q)}{8d(d-1)} + \frac{p(d-p)q(d-q)}{2d^2(d-1)^2} + \frac{p(d-q) + q(d-p)}{24d^2}; \\ & \left. \frac{1}{60} - \frac{p(d-q) + q(d-p)}{24d^2} \right). \end{aligned} \quad (4.39)$$

This result is valid for any dimension we are interested in. It is in fact singular only in  $d = 1$ , that we do not wish to consider. The effective action is manifestly invariant under complex conjugation  $p \leftrightarrow q$  and Hodge duality  $p \leftrightarrow d - q$ , and no one of these symmetries is affected by topological issues, as it will be proved soon. For  $q = 0$  (4.39) reproduces the result found in [5], including the scalar field recovered by setting also  $p = 0$

$$\begin{aligned} B_{(p,0)} \rightarrow & \binom{d}{p} \times \left( 1; \frac{1}{6} - \frac{p}{2d}; \frac{1}{180} - \frac{p(d-p)}{24d(d-1)}; -\frac{1}{360} + \frac{p(d-p)}{6d(d-1)} - \frac{p}{24d}; \right. \\ & \left. \frac{1}{72} - \frac{p(d-p)}{8d(d-1)} + \frac{p}{24d}; \frac{1}{60} - \frac{p}{24d} \right), \\ B_{(0,0)} \rightarrow & \left( 1; \frac{1}{6}; \frac{1}{180}; -\frac{1}{360}; \frac{1}{72}; \frac{1}{60} \right). \end{aligned} \quad (4.40)$$

The last model we are interested in describes  $(p, d-p)$  traceless forms with no gauge symmetry. Their partition function is recovered using the second modular measure of (4.11) and yields

$$\begin{aligned} Z_\beta[g] \propto & -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \oint_\gamma \frac{dz}{2\pi iz} \oint_\gamma \frac{dw}{2\pi iw} \frac{1}{(zw)^{p+1}} [(z+1)(w+1)]^d (z-w)^2 \\ & \times \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0) \langle e^{-S_{\text{int}}} \rangle. \end{aligned} \quad (4.41)$$

The Seeley-DeWitt coefficients are readily obtained from (4.17) and (4.41) as

$$\begin{aligned}
\beta_{(p,d-p)} &\rightarrow \binom{d}{p} \frac{(d+1)!}{(p+1)!(d+1-p)!} \times \left(1; -\frac{1}{3} + \frac{p(d-p)}{(d+1)d}; \right. \\
&\frac{1}{180} + \frac{p(d-p)}{12d^2(d+1)(d-1)} [6(p+1)(d+1-p) - d(d+2)]; \\
&-\frac{2}{45} + \frac{p(d-p)}{12d^2(d+1)(d-1)} [d(5d+7) - 12(p+1)(d+1-p)]; \\
&\frac{1}{18} + \frac{p(d-p)}{12d^2(d+1)(d-1)} [6(p+1)(d+1-p) - d(4d+5)]; \\
&\left. -\frac{1}{40} + \frac{p(d-p)}{12d(d+1)} \right), \tag{4.42}
\end{aligned}$$

that are valid in  $d > 1$  and exhibit explicit symmetry under complex conjugation:  $p \rightarrow d-p$ , whereas Hodge duality acts trivially as in the  $\alpha$  theory.

## 5 Dualities

In this section we will study in more depth exact duality relations and the topological mismatches that potentially arise in this context. Before diving into the duality issues, it is useful to study at the non-perturbative level the structure of our effective actions and the relations between the different models.

### 5.1 Anatomy of the effective actions

In order to uncover exact relations between the effective actions, it is fruitful to switch to a mixed formalism, in between the functional and the operatorial ones. Since the hamiltonian  $H$  and the fermion number operators contained in  $J_1^1$  and  $J_2^2$  commute with each other, they are simultaneously diagonalizable, and the Hilbert space can be viewed as a direct sum of subspaces with definite fermion numbers  $N = n$ ,  $\bar{N} = m$ :  $\mathcal{H} = \bigoplus_{n,m=0}^d \mathcal{H}_{n,m}$ . Thus, we decide to leave the modular integrals as they stand, and to recast the rest of the path integral as a quantum mechanical trace over the Hilbert space. To display this, let us focus on the  $A$  model first. We rewrite the partition function (4.3) as

$$\begin{aligned}
Z_{p,q}^A &\propto \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2 \cos \frac{\phi}{2}\right)^{-2} \left(2 \cos \frac{\theta}{2}\right)^{-2} \int_{\mathbb{P}} \mathcal{D}x \mathcal{D}\bar{x} \int_{\mathbb{A}} D\bar{\psi} D\psi e^{-S_A[X, \tilde{G}_A]} \\
&= \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2 \cos \frac{\phi}{2}\right)^{-2} \left(2 \cos \frac{\theta}{2}\right)^{-2} \text{Tr} \left[ e^{i\phi(J_1^1 - s_1)} e^{i\theta(J_2^2 - s_2)} e^{-\beta H} \right] \\
&= \int_0^\infty \frac{d\beta}{\beta} \underbrace{\oint_{\gamma^-} \frac{dz}{2\pi i z} \oint_{\gamma^+} \frac{dw}{2\pi i w} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1 - s_1)} w^{(J_2^2 - s_2)} e^{-\beta H} \right]}_{Z_{p,q}^A(\beta)}, \tag{5.1}
\end{aligned}$$

where  $s_1 = p + 1 - d/2$  and  $s_2 = d/2 - q - 1$ , and where we defined the partition function density in proper time  $\mathcal{Z}(\beta)$ . In this formula we have rewritten the path integral over matter fields as a quantum mechanical trace. In doing so we made explicit the dependence on the Wilson loop variables  $z = e^{i\phi}$  and  $w = e^{i\theta}$ . We recall that the factors  $\frac{z}{(z+1)^2} \frac{w}{(w+1)^2}$  are the contributions of the SUSY superghosts, and  $H$  is the quantum hamiltonian given in section 3. As a last notational issue, as we have mentioned that the Hilbert space can be viewed as a sum of sectors with fixed fermion numbers, we are going to denote by  $t_{n,m}$  the trace of  $e^{-\beta H}$  taken over the Hilbert subspace with fixed  $N = n$  and  $\bar{N} = m$

$$t_{n,m}(\beta) \equiv \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right]. \quad (5.2)$$

Having presented our new notations, we start by analyzing the simplest of our models, namely the  $B_{(p,q)}$  forms.

The  $B$  model does not gauge supersymmetry, hence the factors coming from the superghosts are absent. Moreover, in this case we do not start from curvatures to arrive at the  $B$  forms, and the Chern-Simons couplings read

$$s_1 = p - \frac{d}{2}, \quad s_2 = \frac{d}{2} - q,$$

so that

$$J_1^1 - s_1 = N - p, \quad J_2^2 - s_2 = -\bar{N} + q.$$

The partition function density  $\mathcal{Z}_{p,q}^B(\beta)$  in our present notation reads

$$\begin{aligned} \mathcal{Z}_{p,q}^B(\beta) &= \oint_{\gamma} \frac{dz}{2\pi i z} \oint_{\gamma} \frac{dw}{2\pi i w} \text{Tr} \left[ z^{(J_1^1 - s_1)} w^{(J_2^2 - s_2)} e^{-\beta H} \right] \\ &= \oint_{\gamma} \frac{dz}{2\pi i z} \oint_{\gamma} \frac{dw}{2\pi i w} \text{Tr} \left[ z^{(N-p)} w^{(-\bar{N}+q)} e^{-\beta H} \right] \\ &= \oint_{\gamma} \frac{dz}{2\pi i z} \oint_{\gamma} \frac{dw}{2\pi i w} \sum_{n,m=0}^d z^{n-p} w^{q-m} \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right] = t_{p,q}(\beta). \end{aligned} \quad (5.3)$$

We can see from (5.3) that, not having gauged the supersymmetries, no pole can arise in  $z = -1$  and  $w = -1$  at the non-perturbative level, and this justifies the use of the unregulated contour  $\gamma$ . The result is given indeed by the single residue at  $z = w = 0$ , and the effective action for  $(p,q)$ -forms obeying just a Laplace-type equation  $\Delta B_{(p,q)} = 0$ , is provided by the single contribution of the subspace  $\mathcal{H}_{p,q}$

$$\mathcal{Z}_{p,q}^B(\beta) = t_{p,q}(\beta). \quad (5.4)$$

We can now examine the structure of the effective action for the  $\beta_{(p,d-p)}$  forms, that obey a null trace condition  $\text{Tr} \beta_{(p,d-p)} = 0$  besides the Laplace-type equation. By using the

same procedure of (5.3) we get

$$\begin{aligned}
\mathcal{Z}_{p,d-p}^\beta(\beta) &= -\frac{1}{2} \oint_{\gamma} \frac{dz}{2\pi iz} \oint_{\gamma} \frac{dw}{2\pi iw} \frac{(z-w)^2}{zw} \text{Tr} \left[ z^{(J_1^1-s)} w^{(J_2^2-s)} e^{-\beta H} \right] \\
&= -\frac{1}{2} \oint_{\gamma} \frac{dz}{2\pi iz} \oint_{\gamma} \frac{dw}{2\pi iw} \frac{(z-w)^2}{zw} \text{Tr} \left[ z^{(N-p)} w^{(-\bar{N}+d-p)} e^{-\beta H} \right] \\
&= -\frac{1}{2} \oint_{\gamma} \frac{dz}{2\pi iz} \oint_{\gamma} \frac{dw}{2\pi iw} \frac{(z-w)^2}{zw} \sum_{n,m=0}^d z^{n-p} w^{d-p-m} \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right] \\
&= t_{p,d-p}(\beta) - \frac{1}{2} t_{p-1,d-p-1}(\beta) - \frac{1}{2} t_{p+1,d-p+1}(\beta) .
\end{aligned} \tag{5.5}$$

As it is natural, one has to subtract from the contribution of  $(p, d-p)$ -forms one term referring to the trace. The other one is effectively the same, since we will show in a while that  $t_{p,q} = t_{d-q,d-p}$ . However, its presence in the form  $t_{p+1,d-p+1}$ , that will be crucial in examining  $\alpha$  forms, is due to the dual constraints we have in Dirac quantization:  $J_2^1 = \text{Tr}$  and  $J_1^2 = g\wedge$ , that are equivalent up to topological mismatches. In the present theory no topological issues arise, indeed  $t_{p+1,d-p+1} = t_{p-1,d-p-1}$ , and one finally has

$$\mathcal{Z}_{p,d-p}^\beta(\beta) = t_{p,d-p}(\beta) - t_{p-1,d-p-1}(\beta) = \mathcal{Z}_{p,d-p}^B(\beta) - \mathcal{Z}_{p-1,d-p-1}^B(\beta) . \tag{5.6}$$

We are now ready to deal with the more complicated models, *i.e.*  $A$  and  $\alpha$ , where supersymmetry is gauged. Here, as shown in (5.1), the additional poles at  $z = -1$  and  $w = -1$  do arise, and one has to use the regulated contours discussed in the previous section. Hence, by taking into account the residue at  $z = 0$  and both residues at  $w = 0$  and  $w = -1$ , one obtains for  $A_{(p,q)}$  forms

$$\begin{aligned}
\mathcal{Z}_{p,q}^A(\beta) &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1-s_1)} w^{(J_2^2-s_2)} e^{-\beta H} \right] \\
&= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(N-p-1)} w^{(-\bar{N}+q+1)} e^{-\beta H} \right] \\
&= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \sum_{n,m=0}^d z^{n-p-1} w^{q+1-m} \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right] \\
&= \sum_{n=0}^p \sum_{m=0}^q (-)^{p-n} (-)^{q-m} (p+1-n)(q+1-m) t_{n,m}(\beta) ,
\end{aligned} \tag{5.7}$$

where we recall that  $s_1 = p+1-d/2$  and  $s_2 = d/2-q-1$ . The structure of the effective action for  $A_{(p,q)}$  forms, dictated by gauge symmetry, is then displayed in terms of the building blocks  $t_{n,m} = \mathcal{Z}_{n,m}^B$

$$\mathcal{Z}_{p,q}^A(\beta) = \sum_{n=0}^p \sum_{m=0}^q (-)^{n+m} (n+1)(m+1) \mathcal{Z}_{p-n,q-m}^B(\beta) , \tag{5.8}$$

where we changed variables  $n \rightarrow p-n$  and  $m \rightarrow q-m$  to cast it in this simple form.

The last model we are going to deal with concerns the  $\alpha$   $(p, d-2-p)$ -forms. Their field equations are analogous, apart from the trace term, to the equations for  $A_{(p,q)}$  forms, and indeed the effective action is recast neatly in terms of the  $Z^A$

$$\begin{aligned}
\mathcal{Z}_{p,d-2-p}^\alpha(\beta) &= -\frac{1}{2} \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \frac{(z-w)^2}{zw} \text{Tr} \left[ z^{(J_1^1-s)} w^{(J_2^2-s)} e^{-\beta H} \right] \\
&= -\frac{1}{2} \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \frac{(z-w)^2}{zw} \text{Tr} \left[ z^{(N-p-1)} w^{(-\bar{N}+d-p-1)} e^{-\beta H} \right] \\
&= -\frac{1}{2} \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \frac{(z-w)^2}{zw} \sum_{n,m=0}^d z^{n-p-1} w^{d-p-1-m} \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right] \\
&= \sum_{n=0}^p \sum_{m=0}^{d-2-p} (-)^{p-n} (-)^{d-p-m} (p+1-n)(d-p-1-m) t_{n,m}(\beta) \\
&\quad - \frac{1}{2} \sum_{n=0}^{p-1} \sum_{m=0}^{d-3-p} (-)^{p-1-n} (-)^{d-p-3-m} (p-n)(d-p-2-m) t_{n,m}(\beta) \\
&\quad - \frac{1}{2} \sum_{n=0}^{p+1} \sum_{m=0}^{d-1-p} (-)^{p+1-n} (-)^{d-p-1-m} (p+1-n)(d-p-m) t_{n,m}(\beta) .
\end{aligned} \tag{5.9}$$

By comparing with (5.7), we immediately recover from the last lines the effective actions of the  $A_{(p,q)}$  forms, in terms of which the structure of  $\mathcal{Z}^\alpha$  becomes apparent

$$\mathcal{Z}_{p,d-2-p}^\alpha(\beta) = \mathcal{Z}_{p,d-2-p}^A(\beta) - \frac{1}{2} \mathcal{Z}_{p-1,d-3-p}^A(\beta) - \frac{1}{2} \mathcal{Z}_{p+1,d-1-p}^A(\beta) . \tag{5.10}$$

As in the case of  $\beta$  forms, we see that the effective action is given by the corresponding contribution of the model without trace constraints *i.e.*  $\mathcal{Z}_{p,d-2-p}^A$ , to which one has to subtract the terms corresponding to the trace and its dual. In this context, however, Hodge duality is affected by topological mismatches, so that  $\mathcal{Z}_{p-1,d-3-p}^A \neq \mathcal{Z}_{p+1,d-1-p}^A$  and no further simplifications occur. This justifies the fact, already discussed in the previous section, that  $\mathcal{Z}_{0,d-2}^\alpha$  differs from the naive expectation of being identical to  $\mathcal{Z}_{0,d-2}^A$ : the difference is given by a topological quantity

$$\mathcal{Z}_{0,d-2}^\alpha(\beta) = \mathcal{Z}_{0,d-2}^A(\beta) - \frac{1}{2} \mathcal{Z}_{1,d-1}^{A \text{ top}}(\beta) . \tag{5.11}$$

## 5.2 Dualities and topological mismatches

Having analyzed the structure of our effective actions in terms of the building blocks  $t_{n,m}(\beta) \equiv \mathcal{Z}_{n,m}^B(\beta)$ , we can now turn to study exact duality relations in the four models at hand. It will turn out that the symmetry under complex conjugation is always preserved, thanks to the choice of regulated contours  $\gamma^-$  and  $\gamma^+$ . On the other hand, invariance under Hodge duality suffers a topological mismatch in the  $A$  model, though it is preserved in the  $B$  model. The remaining cases,  $\alpha$  and  $\beta$ , do not show any mismatch, since Hodge duality acts trivially and does not even transform the fields. Hence, let us focus on the  $A$  and  $B$  forms.

We recall that the operators determining the form degree are given by

$$J_1^1 = \frac{1}{2}[\psi_1^\mu, \bar{\psi}_\mu^1] = N - \frac{d}{2}, \quad J_2^2 = \frac{1}{2}[\psi_2^\mu, \bar{\psi}_\mu^2] = -\bar{N} + \frac{d}{2} \quad (5.12)$$

and that the Chern-Simons couplings differ for the  $A$  and the  $B$  models

$$\begin{aligned} s_1 &= p + 1 - \frac{d}{2}, \quad s_2 = \frac{d}{2} - q - 1, \quad A_{(p,q)} \text{ forms} \\ s_1 &= p - \frac{d}{2}, \quad s_2 = \frac{d}{2} - q, \quad B_{(p,q)} \text{ forms.} \end{aligned} \quad (5.13)$$

It is useful for the present analysis to recall also the gauge fixed fermionic action, that is common to the two models and reads

$$S_{\text{fermion}}^{\text{gf}} = \frac{1}{\beta} \int_0^1 d\tau \left[ \bar{\psi}_\mu^i D_\tau \psi_i^\mu - \frac{1}{2} R_{\mu\bar{\nu}\lambda\bar{\sigma}} \psi^\mu \cdot \bar{\psi}^{\bar{\nu}} \psi^\lambda \cdot \bar{\psi}^{\bar{\sigma}} + i\phi(\bar{\psi}_\mu^1 \psi_1^\mu + s_1) + i\theta(\bar{\psi}_\mu^2 \psi_2^\mu + s_2) \right]. \quad (5.14)$$

We start with the simpler  $B_{(p,q)}$  forms, where one has  $J_1^1 - s_1 = N - p$  and  $J_2^2 - s_2 = -\bar{N} + q$ . Under complex conjugation (c.c.) one has to swap holomorphic and anti-holomorphic indices, effectively exchanging  $p \leftrightarrow q$ . To realize this in the worldline theory it is necessary to exchange the Chern-Simons couplings:  $s_1 \xrightarrow{\text{c.c.}} -s_2$ , as it is evident from (5.13). By looking at the action (5.14) it is clear that transforming the variables as  $\psi_1^\mu \leftrightarrow \bar{\psi}_\mu^2$ ,  $\bar{\psi}_\mu^1 \leftrightarrow \psi_2^\mu$  and  $\phi \leftrightarrow -\theta$ , one brings the action back to its original form. This is perfectly legitimate, since it amounts to a change of dummy integration variables in the path integral. In the mixed formalism we are employing, the transformation  $\psi_1^\mu \leftrightarrow \bar{\psi}_\mu^2$ ,  $\bar{\psi}_\mu^1 \leftrightarrow \psi_2^\mu$  produces  $J_1^1 \leftrightarrow -J_2^2$ , and  $\phi \leftrightarrow -\theta$  corresponds to  $z \leftrightarrow \frac{1}{w}$ . At the level of the effective action, the steps are as follows

$$\begin{aligned} \mathcal{Z}_{q,p}^B(\beta) &= \oint_\gamma \frac{dz}{2\pi iz} \oint_\gamma \frac{dw}{2\pi iw} \text{Tr} \left[ z^{(J_1^1 + s_2)} w^{(J_2^2 + s_1)} e^{-\beta H} \right] \\ &= \oint_\gamma \frac{dz}{2\pi iz} \oint_\gamma \frac{dw}{2\pi iw} \text{Tr} \left[ z^{(-J_2^2 + s_2)} w^{(-J_1^1 + s_1)} e^{-\beta H} \right] \\ &= \oint_\gamma \frac{dw'}{2\pi iw'} \oint_\gamma \frac{dz'}{2\pi iz'} \text{Tr} \left[ w'^{(J_2^2 - s_2)} z'^{(J_1^1 - s_1)} e^{-\beta H} \right] = \mathcal{Z}_{p,q}^B(\beta). \end{aligned} \quad (5.15)$$

In the first line we wrote the effective action for the complex conjugated  $(q, p)$ -form, *i.e.* with couplings  $(-s_2, -s_1)$  instead of  $(s_1, s_2)$ ; in the second step we renamed the fermionic variables, producing  $J_1^1 \leftrightarrow -J_2^2$ , and in the last one we performed the change of variables  $z = \frac{1}{w'}$ ,  $w = \frac{1}{z'}$ , that finally brought the effective action back to the original form, thus proving the symmetry under complex conjugation.

To analyze the effects of Hodge duality on  $B_{(p,q)}$  we proceed along similar lines. The Hodge transformation tells us to exchange  $p \leftrightarrow d - q$ , that in terms of Chern-Simons couplings is realized by  $s_1 \xrightarrow{*} s_2$ . In order to undo the exchange and prove the symmetry one has to transform  $\psi_1^\mu \leftrightarrow \psi_2^\mu$ ,  $\bar{\psi}_\mu^1 \leftrightarrow \bar{\psi}_\mu^2$  and  $\phi \leftrightarrow \theta$ , that is  $z \leftrightarrow w$ . This sends  $J_1^1 \leftrightarrow J_2^2$ ,

and one can see the invariance of the effective action

$$\begin{aligned}\mathcal{Z}_{d-q,d-p}^B(\beta) &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^-} \frac{dw}{2\pi iw} \text{Tr} \left[ z^{(J_1^1-s_2)} w^{(J_2^2-s_1)} e^{-\beta H} \right] \\ &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^-} \frac{dw}{2\pi iw} \text{Tr} \left[ z^{(J_2^2-s_2)} w^{(J_1^1-s_1)} e^{-\beta H} \right] = \mathcal{Z}_{p,q}^B(\beta) .\end{aligned}\tag{5.16}$$

Again, we wrote first the Hodge dual effective action by using the coupling  $(s_2, s_1)$  instead of  $(s_1, s_2)$ ; then we performed the exchange between fermionic species, sending  $J_1^1 \leftrightarrow J_2^2$ , and obtained immediately the original expression, due to the explicit symmetry between the  $z$  and  $w$  integrals. This last symmetry will be missing when dealing with the  $A$  model. This argument proves the invariance of the effective action under Hodge duality. Together with complex conjugation it ensures the exact validity of the following identities

$$\begin{aligned}\mathcal{Z}_{p,q}^B(\beta) &= \mathcal{Z}_{q,p}^B(\beta) = \mathcal{Z}_{d-q,d-p}^B(\beta) = \mathcal{Z}_{d-p,d-q}^B(\beta) , \quad \text{or} \\ t_{n,m}(\beta) &= t_{m,n}(\beta) = t_{d-m,d-n}(\beta) = t_{d-n,d-m}(\beta) .\end{aligned}\tag{5.17}$$

We are now going to investigate the more interesting case of  $(p, q)$ -form gauge fields  $A_{(p,q)}$  where, due to the different Chern-Simons couplings, one has  $J_1^1 - s_1 = N - p - 1$  and  $J_2^2 - s_2 = -\bar{N} + q + 1$ . We shall study first the action of complex conjugation. The transformed effective action is written by using the Chern-Simons couplings  $(-s_2, -s_1)$ , and the change of variables to undo the transformation is the same as before

$$\begin{aligned}\mathcal{Z}_{q,p}^A(\beta) &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1+s_2)} w^{(J_2^2+s_1)} e^{-\beta H} \right] \\ &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(-J_2^2+s_2)} w^{(-J_1^1+s_1)} e^{-\beta H} \right] \\ &= \oint_{\gamma^+} \frac{dw'}{2\pi iw'} \oint_{\gamma^-} \frac{dz'}{2\pi iz'} \frac{w'}{(w'+1)^2} \frac{z'}{(z'+1)^2} \text{Tr} \left[ w'^{(J_2^2-s_2)} z'^{(J_1^1-s_1)} e^{-\beta H} \right] = \mathcal{Z}_{p,q}^A(\beta) .\end{aligned}\tag{5.18}$$

The important difference from the  $B$  form computation is the presence of regulated contours  $\gamma^-$  and  $\gamma^+$ : when performing the change of variables  $z = \frac{1}{w'}$ ,  $w = \frac{1}{z'}$ , each regulated contour maps to the other one  $\gamma^- \leftrightarrow \gamma^+$ . In this circumstance the new integrals coincide with the original form of the effective action, and no mismatch appears, proving invariance under complex conjugation

$$\mathcal{Z}_{p,q}^A(\beta) = \mathcal{Z}_{q,p}^A(\beta) .\tag{5.19}$$

Alternatively, this could have been proved using the manifest symmetry under  $p \leftrightarrow q$  of (5.7) and (5.17).

The action of Hodge duality is considerably more involved: the dual effective action is written with couplings  $(s_2, s_1)$ , but the regulated contours will be in the wrong order and

a mismatch appears. Explicitly one has

$$\begin{aligned}
\mathcal{Z}_{d-2-q,d-2-p}^A(\beta) &= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1-s_2)} w^{(J_2^2-s_1)} e^{-\beta H} \right] \\
&= \oint_{\gamma^-} \frac{dz}{2\pi iz} \oint_{\gamma^+} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_2^2-s_2)} w^{(J_1^1-s_1)} e^{-\beta H} \right] \\
&= \oint_{\gamma^+} \frac{dz'}{2\pi iz'} \oint_{\gamma^-} \frac{dw'}{2\pi iw'} \frac{z'}{(z'+1)^2} \frac{w'}{(w'+1)^2} \text{Tr} \left[ z'^{(J_1^1-s_1)} w'^{(J_2^2-s_2)} e^{-\beta H} \right]
\end{aligned} \tag{5.20}$$

where in the last line we simply renamed  $z = w'$  and  $w = z'$  for an immediate legibility. Hence we see that the new effective action is identical to the original one in (5.7) apart from the contour choice, that is inverted. Let us denote with  $\gamma^0$  a small circle surrounding  $-1$  in the complex  $z$  or  $w$  plane, so that  $\gamma^+ = \gamma^- + \gamma^0$ . The mismatch between the two effective actions is then given by

$$\begin{aligned}
\mathcal{Z}_{d-2-q,d-2-p}^A(\beta) &= \oint_{\gamma^+} \frac{dz}{2\pi iz} \oint_{\gamma^-} \frac{dw}{2\pi iw} \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1-s_1)} w^{(J_2^2-s_2)} e^{-\beta H} \right] \\
&= \mathcal{Z}_{p,q}^A(\beta) + \left[ \oint_{\gamma^0} \oint_{\gamma^+} - \oint_{\gamma^-} \oint_{\gamma^0} - \oint_{\gamma^0} \oint_{\gamma^0} \right] \frac{dz}{2\pi iz} \frac{dw}{2\pi iw} \\
&\quad \times \frac{z}{(z+1)^2} \frac{w}{(w+1)^2} \text{Tr} \left[ z^{(J_1^1-s_1)} w^{(J_2^2-s_2)} e^{-\beta H} \right] \\
&= \mathcal{Z}_{p,q}^A(\beta) + \left[ \sum_{n,m=0}^d - \sum_{n=0}^p \sum_{m=0}^d - \sum_{n=0}^d \sum_{m=0}^q \right] f_{n,m}(p, q; \beta),
\end{aligned} \tag{5.21}$$

where we defined

$$f_{n,m}(p, q; \beta) \equiv (-)^{n-p} (-)^{m-q} (p+1-n)(q+1-m) t_{n,m}(\beta). \tag{5.22}$$

To obtain the last line we computed the residues as dictated by the contours, recalling that  $\text{Tr} \left[ z^{(J_1^1-s_1)} w^{(J_2^2-s_2)} e^{-\beta H} \right] = \sum_{n,m=0}^d z^{n-p-1} w^{-m+q+1} t_{n,m}$ . We shall demonstrate in Appendix C that the difference  $\mathcal{Z}_{d-2-q,d-2-p}^A(\beta) - \mathcal{Z}_{p,q}^A(\beta)$  is purely topological, indeed given by linear combinations of quantum mechanical indices of the form  $\text{Tr}(-1)^F$  and  $\text{Tr}(-1)^F F$ , where  $F$  is a suitable fermion number operator. The main identifications we need, to be proven in Appendix C, are

$$\begin{aligned}
\sum_{m=0}^d (-)^m t_{n,m}(\beta) &= \text{Tr}_{\mathcal{H}_n}(-1)^{\bar{N}} = \text{ind}(\Omega^{n,0}, \bar{\partial}) , \quad \text{and complex conjugate} \\
\sum_{n,m=0}^d (-)^{n+m} t_{n,m}(\beta) &= \text{Tr}_{\mathcal{H}}(-1)^{N+\bar{N}} = \chi(\mathcal{M}) ,
\end{aligned} \tag{5.23}$$

where  $\mathcal{H}_n = \bigoplus_{m=0}^d \mathcal{H}_{n,m}$  is the Hilbert subspace with fixed  $N = n$ ,  $\chi(\mathcal{M})$  is the Euler characteristics of our manifold, and  $\text{ind}(\Omega^{n,0}, \bar{\partial})$  denotes the Dolbeault index twisted by the



holomorphic vector bundle of  $(n, 0)$ -forms. By putting together the various contributions from (5.21), and using (5.23), we finally get the relation between the effective actions of Hodge dual forms

$$\begin{aligned}
\mathcal{Z}_{d-2-q, d-2-p}^A(\beta) - \mathcal{Z}_{p,q}^A(\beta) &= (-)^{q+d} \mathcal{Z}_{p, d-1}^{A \text{ top}}(\beta) + (-)^{p+d} \mathcal{Z}_{d-1, q}^{A \text{ top}}(\beta) + (-)^{p+q} \mathcal{Z}_{d-1, d-1}^{A \text{ top}}(\beta) \\
&+ (d-1-p)(-)^{p+q} \sum_{m=0}^q (-)^m (q+1-m) \text{ind}(\Omega^{m,0}, \bar{\partial}) \\
&+ (d-1-q)(-)^{p+q} \sum_{n=0}^p (-)^n (p+1-n) \text{ind}(\Omega^{n,0}, \bar{\partial}) \\
&+ (-)^{p+q} \left[ \left(p+1-\frac{d}{2}\right) \left(q+1-\frac{d}{2}\right) - \frac{d^2}{4} \right] \chi(\mathcal{M})
\end{aligned} \tag{5.24}$$

where the contributions of non-propagating top forms (*i.e.* forms with  $p$  or  $q$  exceeding  $d-2$ ) have been read down from the explicit formula (5.7) for  $p = d-1$  and/or  $q = d-1$ . It may appear rather obscure that such top forms, although obviously not propagating, are in fact topological. To render it manifest, we stress that they have a genuine topological interpretation, since they coincide with analytic Ray-Singer torsions of the  $\bar{\partial}$ -operator [40], as it will be shown in Appendix C, namely

$$\begin{aligned}
\int_0^\infty \frac{d\beta}{\beta} \sum_{m=0}^d (-)^m m t_{n,m}(\beta) &= \int_0^\infty \frac{d\beta}{\beta} \text{Tr}_{\mathcal{H}_n} \left[ (-1)^{\bar{N}} \bar{N} e^{-\beta H} \right] = 2 \ln T_n(\mathcal{M}) \quad \Rightarrow \\
Z_{p, d-1}^{A \text{ top}} &= 2 \sum_{n=0}^p (-)^{n+1} (n+1) \ln T_{d-p+n}(\mathcal{M}) ,
\end{aligned} \tag{5.25}$$

where  $T_n(\mathcal{M})$  denotes the  $\bar{\partial}$ -torsion obtained by summing in  $m$  the contributions of  $(n, m)$ -forms at fixed  $n$ . We stress here that, unlike the case of Dolbeault indices, the identification with the Ray-Singer torsions is valid at the level of effective actions  $Z$  and not of densities  $\mathcal{Z}$ . A similar phenomenon is present for ordinary differential forms [41].

These topological mismatches, that arise as obstructions in defining uniquely the Hodge dual gauge fields on the whole manifold, are visible in our computations only in  $d = 3$  and  $d = 2$ , since in higher dimensions they appear in higher order Seeley-DeWitt coefficients. Nevertheless, also in these particular cases we can address very non-trivial checks. For instance, in  $d = 3$  we can check (5.24) for  $p = q = 0$  and for  $p = 0, q = 1$ :

$$\begin{aligned}
d &= 3 \\
\mathcal{Z}_{1,1}^A(\beta) - \mathcal{Z}_{0,0}^A(\beta) &= \mathcal{Z}_{2,2}^{A \text{ top}}(\beta) - 2 \mathcal{Z}_{2,0}^{A \text{ top}}(\beta) \\
\mathcal{Z}_{2,0}^{A \text{ top}}(\beta) - \mathcal{Z}_{2,2}^{A \text{ top}}(\beta) - \mathcal{Z}_{2,1}^{A \text{ top}}(\beta) &= 0 ,
\end{aligned} \tag{5.26}$$

where we have omitted the index contributions since they are of order  $\beta^3$ . From (4.23) we

can read off the SDW coefficients of  $A_{(0,0)}$  and  $A_{(1,1)}$ . Their difference yields

$$d = 3$$

$$A_{(1,1)} - A_{(0,0)} \rightarrow \left(0; 0; \frac{5}{12}; -\frac{2}{3}; \frac{1}{4}; 0\right), \quad (5.27)$$

while from (4.19) we find the contributions of the top forms as

$$d = 3$$

$$A_{(2,2)} \rightarrow \left(0; 0; \frac{1}{2}; -1; \frac{1}{2}; 0\right),$$

$$A_{(2,0)} \rightarrow \left(0; 0; \frac{1}{24}; -\frac{1}{6}; \frac{1}{8}; 0\right), \quad (5.28)$$

$$A_{(2,1)} \rightarrow \left(0; 0; -\frac{11}{24}; \frac{5}{6}; -\frac{3}{8}; 0\right),$$

and it is straightforward to see that they satisfy both relations in (5.26).

In two complex dimensions we can check (5.24) at  $p = q = 0$ , for which we get the following relation between topological quantities

$$d = 2$$

$$\mathcal{Z}_{1,1}^{A \text{ top}}(\beta) + 2\mathcal{Z}_{1,0}^{A \text{ top}}(\beta) + 2 \text{ind}(\bar{\partial}) - \chi(\mathcal{M}) = 0. \quad (5.29)$$

Again, the top form contributions can be computed from (4.19):

$$d = 2$$

$$A_{(1,1)} \rightarrow \left(0; 1; \frac{1}{2}; -\frac{11}{12}; \frac{5}{12}; \frac{1}{12}\right), \quad (5.30)$$

$$A_{(1,0)} \rightarrow \left(0; -\frac{1}{2}; -\frac{1}{24}; \frac{1}{8}; -\frac{1}{12}; -\frac{1}{24}\right),$$

and they give, as sum of the effective action densities,

$$\mathcal{Z}_{1,1}^{A \text{ top}}(\beta) + 2\mathcal{Z}_{1,0}^{A \text{ top}}(\beta) = \frac{1}{4\pi^2} \int d^2x_0 d^2\bar{x}_0 g(x_0) \left[ \frac{5}{12} R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} - \frac{2}{3} R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + \frac{1}{4} R^2 \right]. \quad (5.31)$$

The index of the Dolbeault operator is given by (C.15) and in components reads

$$\text{ind}(\bar{\partial}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} \left[ \frac{1}{3} \text{tr} \mathcal{R} \wedge \mathcal{R} - \text{tr} \mathcal{R} \wedge \text{tr} \mathcal{R} \right]$$

$$= \frac{1}{32\pi^2} \int d^2x_0 d^2\bar{x}_0 g(x_0) \left[ \frac{1}{3} R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} - \frac{4}{3} R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + R^2 \right], \quad (5.32)$$

with the Kähler curvature two-form being  $\mathcal{R}^\lambda{}_\sigma = R_{\mu\bar{\nu}}{}^\lambda{}_\sigma dx^\mu \wedge d\bar{x}^{\bar{\nu}}$ . The Euler characteristics is given by

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} \left[ \epsilon^{ABCD} \mathcal{R}_{AB}^{(G)} \wedge \mathcal{R}_{CD}^{(G)} \right]$$

$$= \frac{1}{8\pi^2} \int d^2x_0 d^2\bar{x}_0 g(x_0) \left[ R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} - 2 R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + R^2 \right], \quad (5.33)$$

where  $\mathcal{R}_{AB}^{(G)}$  denotes the curvature two-form built from the riemannian metric  $G_{MN}$ , and  $A, B, ..$  riemannian  $SO(2d)$  flat indices. Putting together (5.31), (5.32) and (5.33), one easily verifies that the relation (5.29) is indeed satisfied.

## 6 Conclusions

We have described several quantum theories of massless  $(p, q)$ -forms, with and without gauge symmetries. We have employed a worldline approach that makes use of a  $U(2)$  spinning particle, which is identified by a supersymmetric nonlinear sigma model with four supercharges suitably gauged. In particular, we have studied effective actions of massless  $(p, q)$ -forms on arbitrary Kähler spaces, and studied duality relations. For the A-type models they include topological mismatches, related to index densities and analytic torsions. Calculation of several heat kernel coefficients has been presented as well. The physical motivations for studying such models are indirect, since a honest spacetime interpretation is prevented by the complex nature of the target space (also choosing an appropriate signature, time must necessarily be complex to preserve the complex structure of the target manifold). Nevertheless complex manifolds find useful applications in the context of string and/or supersymmetric theories. For sure, they offer a useful arena to test methods of quantum field theory, including worldline approaches to theories on curved backgrounds and ideas from the current studies of higher spin fields.

## Acknowledgments

We wish to thank Roberto Zucchini for valuable discussions. The work of FB was supported in part by the MIUR-PRIN contract 2009-KHZKRX. C.I. wishes to thank the Erwin Schrödinger Institute in Vienna for kind hospitality during the final stage of this work, and gratefully acknowledges the “Università degli Studi G. Marconi” in Rome for partial support.

## A Notations and conventions

We list here the conventions and formulas of Kähler geometry we make use of in the main text. When needed, we rewrite some geometric quantities by using both holomorphic coordinates  $(x^\mu, \bar{x}^{\bar{\mu}})$ ,  $\mu, \bar{\mu} = 1, \dots, d$ , and riemannian coordinates  $X^M$ , with  $M = 1, \dots, 2d$ . We deal with Kähler manifolds  $\mathcal{M}$  with  $d$  complex dimensions, whose metric is specified by

$$ds^2 = G_{MN} dX^M dX^N = 2g_{\mu\bar{\nu}} dx^\mu d\bar{x}^{\bar{\nu}}, \quad (\text{A.1})$$

while the integration measure reads

$$d\mu = \sqrt{\det G_{MN}} d^D X = \det g_{\mu\bar{\nu}} d^d x d^d \bar{x} \quad (\text{A.2})$$

with the convention

$$d^d x d^d \bar{x} \equiv i^d \prod_{\mu=1}^d dx^\mu \wedge d\bar{x}^{\bar{\mu}}. \quad (\text{A.3})$$

We also use the notation  $g \equiv \det g_{\mu\bar{\nu}}$  for the determinant of the Kähler metric. On flat manifolds one can employ cartesian coordinates for which  $G_{MN} = \delta_{MN}$  and  $g_{\mu\bar{\nu}} = \delta_{\mu\bar{\nu}}$ . We relate real and complex coordinates by

$$x^\mu = \frac{1}{\sqrt{2}}(X^{2\mu-1} + iX^{2\mu}), \quad \bar{x}^{\bar{\mu}} = \frac{1}{\sqrt{2}}(X^{2\mu-1} - iX^{2\mu}), \quad \mu = 1, \dots, d. \quad (\text{A.4})$$

We come now to connections and curvatures. In holomorphic coordinates the nonzero Christoffel symbols are given, in terms of the metric, by

$$\Gamma_{\nu\lambda}^\mu = g^{\mu\bar{\mu}} \partial_\nu g_{\lambda\bar{\mu}}, \quad \Gamma_{\bar{\nu}\bar{\lambda}}^{\bar{\mu}} = g^{\mu\bar{\mu}} \partial_{\bar{\nu}} g_{\lambda\bar{\mu}}, \quad (\text{A.5})$$

where  $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$  and  $\partial_{\bar{\mu}} \equiv \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ . The nonzero components of the Riemann tensor read

$$R^\mu{}_{\nu\bar{\sigma}\lambda} = \partial_{\bar{\sigma}} \Gamma_{\nu\lambda}^\mu, \quad R^{\bar{\mu}}{}_{\bar{\nu}\sigma\bar{\lambda}} = \partial_\sigma \Gamma_{\bar{\nu}\bar{\lambda}}^{\bar{\mu}}, \quad (\text{A.6})$$

while we define the Ricci tensor and the curvature scalar as

$$R_{\mu\bar{\nu}} = -R^\lambda{}_{\lambda\bar{\nu}\mu} = -\partial_\mu \bar{\Gamma}_{\bar{\nu}} = -\partial_{\bar{\nu}} \Gamma_\mu = -\partial_\mu \partial_{\bar{\nu}} \ln g, \quad (\text{A.7})$$

$$R = g^{\mu\bar{\nu}} R_{\mu\bar{\nu}}.$$

The curvature two-forms for Kähler and riemannian connections is denoted by

$$\mathcal{R}^\mu{}_\nu = R_{\lambda\bar{\sigma}}{}^\mu{}_\nu dx^\lambda \wedge d\bar{x}^{\bar{\sigma}}, \quad \mathcal{R}_{AB}^{(G)} = \frac{1}{2} R_{MNAB} dX^M \wedge dX^N, \quad (\text{A.8})$$

and the volume forms for Kähler and riemannian metrics read

$$g(x, \bar{x}) \epsilon_{\mu_1 \dots \mu_d} \epsilon_{\bar{\nu}_1 \dots \bar{\nu}_d}, \quad \sqrt{G(X)} \epsilon_{M_1 \dots M_{2d}}. \quad (\text{A.9})$$

The  $(p, q)$ -forms are written in components as

$$A_{(p,q)} = A_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q}(x, \bar{x}) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge d\bar{x}^{\bar{\nu}_1} \wedge \dots \wedge d\bar{x}^{\bar{\nu}_q}, \quad (\text{A.10})$$

but for convenience of notation, from now on, the wedge product is understood whenever a  $dx^\mu$  or  $d\bar{x}^{\bar{\mu}}$  is acting. Number operators, trace operator and multiplication by the Kähler form act as

$$\begin{aligned} N &= dx^\mu \frac{\partial}{\partial(dx^\mu)} , \quad \bar{N} = d\bar{x}^{\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\nu}})} \\ \text{Tr} &= g^{\mu\bar{\nu}} \frac{\partial^2}{\partial(dx^\mu)\partial(d\bar{x}^{\bar{\nu}})} , \quad g\wedge = g_{\mu\bar{\nu}} dx^\mu d\bar{x}^{\bar{\nu}} . \end{aligned} \quad (\text{A.11})$$

In defining Dolbeault operators and their adjoint we choose to denote by  $\partial^\dagger$  and  $\bar{\partial}^\dagger$  the actual adjoints of  $\partial$  and  $\bar{\partial}$ , so that a minus sign appears in the divergences

$$\begin{aligned} \partial &= dx^\mu \partial_\mu , \quad \bar{\partial} = d\bar{x}^{\bar{\mu}} \partial_{\bar{\mu}} \\ \partial^\dagger &= -g^{\mu\bar{\nu}} \frac{\partial}{\partial(dx^\mu)} \nabla_{\bar{\nu}} , \quad \bar{\partial}^\dagger = -g^{\mu\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\nu}})} \nabla_\mu . \end{aligned} \quad (\text{A.12})$$

The  $U(d)$  ‘‘Lorentz’’ generators are given by

$$M^{\mu\bar{\nu}} = g^{\nu\bar{\nu}} dx^\mu \frac{\partial}{\partial(dx^\nu)} - g^{\mu\bar{\mu}} d\bar{x}^{\bar{\nu}} \frac{\partial}{\partial(d\bar{x}^{\bar{\mu}})} . \quad (\text{A.13})$$

We denote by  $\nabla^2$  the curved space laplacian

$$\nabla^2 = G^{MN} \nabla_M \nabla_N = g^{\mu\bar{\nu}} (\nabla_\mu \nabla_{\bar{\nu}} + \nabla_{\bar{\nu}} \nabla_\mu) , \quad (\text{A.14})$$

while we reserve the symbol  $\Delta$  for the full Laplace-Beltrami operator with Dolbeault normalization

$$\Delta = \frac{\nabla^2}{2} + \frac{1}{2} R_{\mu\bar{\nu}\lambda\bar{\sigma}} M^{\mu\bar{\nu}} M^{\lambda\bar{\sigma}} = -(\partial\partial^\dagger + \partial^\dagger\partial) = -(\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}) = -\frac{1}{2} (dd^\dagger + d^\dagger d) \quad (\text{A.15})$$

so that the full quantum hamiltonian acts as  $H = -\Delta$ .

Finally, in order to compare the heat kernel coefficients computed in the present paper with those present in the literature, we relate the quadratic terms in curvatures with riemannian normalization to those with Kähler normalization

$$R_{(G)} \equiv g^{MN} R_{MN} = 2R , \quad R_{MN} R^{MN} = 2R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} , \quad R_{MNRS} R^{MNRS} = 4R_{\mu\bar{\nu}\rho\bar{\sigma}} R^{\mu\bar{\nu}\rho\bar{\sigma}} . \quad (\text{A.16})$$

Note in particular that the curvature scalar with Kähler normalization is one half of the standard riemannian one.

## B Propagators in TS regularization

We list here all the propagators in the continuum limit, along with their derivatives and equal time expressions, as well as their regulated expressions valid in Time Slicing regularization. All the TS prescriptions follow by considering the Dirac distributions acting as Kronecker deltas, and taking into account the regulated value  $\theta(0) = \frac{1}{2}$ .

With  $\tau, \sigma \in [0, 1]$  and denoting  $\phi_i = (\phi, \theta)$  the propagators read

$$\begin{aligned}\Delta(\tau, \sigma) &= \sigma(\tau - 1)\theta(\tau - \sigma) + \tau(\sigma - 1)\theta(\sigma - \tau) , \\ \Delta_{\text{gh}}(\tau, \sigma) &= \delta(\tau, \sigma) , \\ \Delta_{\text{f}}(\tau - \sigma, \phi_i) &= \frac{e^{-i\phi_i(\tau - \sigma)}}{2 \cos \frac{\phi_i}{2}} \left[ e^{i\frac{\phi_i}{2}} \theta(\tau - \sigma) - e^{-i\frac{\phi_i}{2}} \theta(\sigma - \tau) \right] .\end{aligned}\tag{B.1}$$

Derivatives of the bosonic propagator in the continuum limit are

$$\begin{aligned}\bullet\Delta(\tau, \sigma) &= \sigma - \theta(\sigma - \tau) , \quad \Delta^\bullet(\tau, \sigma) = \tau - \theta(\tau - \sigma) , \\ \bullet\Delta^\bullet(\tau, \sigma) &= 1 - \delta(\tau, \sigma) ,\end{aligned}\tag{B.2}$$

where left (right) dots indicate derivatives with respect to the first (second) variable.

In Time Slicing regularization the following relations hold

$$\begin{aligned}\bullet\Delta^\bullet(\tau, \sigma) + \Delta_{\text{gh}}(\tau, \sigma) &= 1 , \\ \Delta_{\text{f}}(\tau - \sigma, \phi_i) \Delta_{\text{f}}(\sigma - \tau, \phi_i) &= -\frac{1}{4} \cos^{-2} \frac{\phi_i}{2} ,\end{aligned}\tag{B.3}$$

the first one ensuring that no products of delta distributions will ever arise in perturbative calculations. Finally, equal time expressions of the propagators and of their derivatives are obtained from the aforementioned TS prescriptions, and given explicitly by

$$\begin{aligned}\Delta(\tau, \tau) &= \tau(\tau - 1) , \quad \bullet\Delta(\tau, \tau) = \Delta^\bullet(\tau, \tau) = \tau - \frac{1}{2} , \\ \Delta_{\text{f}}(0, \phi_i) &= \frac{i}{2} \tan \frac{\phi_i}{2} .\end{aligned}\tag{B.4}$$

## C Topological quantities

We dedicate this appendix to demonstrate the topological identifications we made use of in the main text. Let us start with the Dolbeault indices. We mentioned that our Hilbert space can be viewed as a direct sum of subspaces with fixed fermion numbers

$$\mathcal{H} = \bigoplus_{n,m=0}^d \mathcal{H}_{n,m} .\tag{C.1}$$

Since  $N$ ,  $\bar{N}$  and  $H$  are self-adjoint and commute with each other, they can be diagonalized simultaneously to provide a basis of eigenstates for each subspace  $\mathcal{H}_{n,m}$ . The same is true if one sums over one fermion number, obtaining the Hilbert subspace with only one fermion number fixed

$$\mathcal{H}_n = \bigoplus_{m=0}^d \mathcal{H}_{n,m} .\tag{C.2}$$

We can focus now on one alternate sum of  $t_{n,m}$  coefficients, that can be rewritten as

$$\sum_{m=0}^d (-)^m t_{n,m}(\beta) = \sum_{m=0}^d (-)^m \text{Tr}_{\mathcal{H}_{n,m}} \left[ e^{-\beta H} \right] = \text{Tr}_{\mathcal{H}_n} \left[ (-)^{\bar{N}} e^{-\beta H} \right] .\tag{C.3}$$

We notice that the part of the superalgebra generated by  $Q_2$ ,  $\bar{Q}^2$  and  $H$  acts separately within each  $\mathcal{H}_n$ . The Witten index argument is then applicable, and tells us that for each  $H$ -eigenstate  $|j\rangle$  with energy  $\epsilon_j$ , there is another eigenstate  $Q_2|j\rangle$  with the same energy and opposite value of  $(-)^{\bar{N}}$ , and hence only zero modes of the hamiltonian survive in the trace written above. Since our hamiltonian coincides with the Dolbeault laplacian, the trace effectively counts with alternate signs the numbers of harmonic  $(n, m)$ -forms, *i.e.* the Hodge numbers  $h^{n,m}$

$$\begin{aligned} \sum_{m=0}^d (-)^m t_{n,m}(\beta) &= \text{Tr}_{\mathcal{H}_n} \left[ (-)^{\bar{N}} e^{-\beta H} \right] = \sum_{m=0}^d (-)^m (\text{number of harmonic } (n,m)\text{-forms}) \\ &= \sum_{m=0}^d (-)^m h^{n,m}(\mathcal{M}) , \end{aligned} \tag{C.4}$$

that for  $n = 0$  would be the Dolbeault index:  $\sum_{m=0}^d (-)^m h^{0,m} = \text{ind}(\bar{\partial})$ . The generalization we need requires to consider Dolbeault operators acting on a holomorphic vector bundle  $V$ . The corresponding Hodge numbers provide the index of the twisted Dolbeault operator [42]

$$\sum_{m=0}^d (-)^m h^{0,m}(V, \mathcal{M}) = \text{ind}(V, \bar{\partial}) . \tag{C.5}$$

For our purposes, the role of  $V$  is played by the bundle of  $(n, 0)$ -forms:  $V = \Omega^{n,0}$ , and the Hodge numbers can be identified as  $h^{n,m}(\mathcal{M}) = h^{0,m}(\Omega^{n,0}, \mathcal{M})$ , providing us with the Dolbeault index twisted by  $(n, 0)$ -forms

$$\sum_{m=0}^d (-)^m t_{n,m}(\beta) = \sum_{m=0}^d (-)^m h^{n,m}(\mathcal{M}) = \text{ind}(\Omega^{n,0}, \bar{\partial}) . \tag{C.6}$$

By symmetry under complex conjugation the same result is valid exchanging the roles of  $n$  and  $m$ , and substituting  $\bar{\partial}$  by  $\partial$ . The explicit formula for it is given as usual by the Atiyah-Singer index theorem, that in the present case reduces to the Hirzebruch-Riemann-Roch theorem

$$\text{ind}(V, \bar{\partial}) = \int_{\mathcal{M}} \text{Td}(T\mathcal{M}^+) \text{ch}(V) , \tag{C.7}$$

where  $\text{Td}(T\mathcal{M}^+)$  is the Todd class of the holomorphic tangent space bundle

$$\text{Td}(T\mathcal{M}^+) = \det \left( \frac{i\mathcal{R}/2\pi}{1 - e^{-i\mathcal{R}/2\pi}} \right) , \tag{C.8}$$

with  $\mathcal{R}$  the Kähler curvature form (A.8), and

$$\text{ch}(V) = \text{tr} e^{i\mathcal{F}_V/2\pi} \tag{C.9}$$

the Chern character of the vector bundle  $V$ ,  $\mathcal{F}_V$  being its curvature two-form. Since our  $V$  consists of  $(n, 0)$ -forms, the formula becomes

$$\text{ind}(\Omega^{n,0}, \bar{\partial}) = \int_{\mathcal{M}} \text{Td}(T\mathcal{M}^+) \text{ch} \left( \bigwedge^n T^* \mathcal{M} \right) . \tag{C.10}$$

The Chern character for the anti-symmetric product can be computed by means of curvature eigenvalues  $x_i$  (see [42]) as

$$\text{ch}(T\mathcal{M}) = \sum_i e^{x_i} \rightarrow \text{ch}\left(\bigwedge^n T^*\mathcal{M}\right) = \sum_{i_1 < i_2 \dots < i_n} e^{-x_{i_1}} e^{-x_{i_2}} \dots e^{-x_{i_n}}, \quad (\text{C.11})$$

and can be reexpressed as

$$\begin{aligned} \text{ch}\left(\bigwedge^n T^*\mathcal{M}\right) &= \frac{1}{n!} \left\{ \left(\text{tr} e^{-i\mathcal{R}/2\pi}\right)^n - \binom{n}{2} \left(\text{tr} e^{-i\mathcal{R}/2\pi}\right)^{n-2} \left(\text{tr} e^{-2i\mathcal{R}/2\pi}\right) \right. \\ &\quad \left. + \sum_{k=3}^n \binom{n}{k} \frac{2k-n-1}{n-k+1} \left(\text{tr} e^{-i\mathcal{R}/2\pi}\right)^{n-k} \left(\text{tr} e^{-ki\mathcal{R}/2\pi}\right) \right\}. \end{aligned} \quad (\text{C.12})$$

For a check we are able to perform in section 5 we need  $\text{ind}(\bar{\partial})$  in two complex dimensions

$$\text{ind}(\bar{\partial}) = \int_{\mathcal{M}} \text{Td}(T\mathcal{M}^+). \quad (\text{C.13})$$

The Todd class can be expanded as

$$\text{Td}(T\mathcal{M}^+) = 1 + \frac{1}{2} \text{tr} \left( \frac{i\mathcal{R}}{2\pi} \right) - \frac{1}{24} \text{tr} \left( \frac{i\mathcal{R}}{2\pi} \right)^2 + \frac{1}{8} \left[ \text{tr} \left( \frac{i\mathcal{R}}{2\pi} \right) \right]^2 + \mathcal{O}(\mathcal{R}^3), \quad (\text{C.14})$$

and the index in  $d = 2$  is

$$\text{ind}(\bar{\partial}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} \left[ \frac{1}{3} \text{tr} \mathcal{R} \wedge \mathcal{R} - \text{tr} \mathcal{R} \wedge \text{tr} \mathcal{R} \right]. \quad (\text{C.15})$$

Having proved that  $\sum_{m=0}^d (-)^m t_{n,m}(\beta) = \sum_{m=0}^d (-)^m h^{n,m}(\mathcal{M})$ , it is evident that summing further on  $n$  with an extra factor of  $(-)^n$  provides the alternate sum of the real Betti numbers, yielding the Euler characteristics

$$\sum_{n,m=0}^d (-)^{m+n} t_{n,m}(\beta) = \sum_{n,m=0}^d (-)^{m+n} h^{n,m}(\mathcal{M}) = \sum_{k=0}^{2d} (-)^k b_k(\mathcal{M}) = \text{ind}(d) = \chi(\mathcal{M}), \quad (\text{C.16})$$

which is given as the integral of the Euler class  $e(\mathcal{M})$ , defined as the Pfaffian of the riemannian curvature

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} e(\mathcal{M}) = \int_{\mathcal{M}} \text{Pf} \left( \frac{\mathcal{R}^{(G)}}{2\pi} \right). \quad (\text{C.17})$$

Again we can check our formulas in  $d = 2$ , where it gives

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} \left[ \epsilon^{ABCD} \mathcal{R}_{AB}^{(G)} \wedge \mathcal{R}_{CD}^{(G)} \right], \quad (\text{C.18})$$

with  $A, B, \dots$  being flat  $SO(2d)$  indices.



There is still another useful identity that we need in section 5 to prove the mismatch formula (5.24). Let us consider the sum  $\sum_{n,m=0}^d (-)^{n+m} n t_{n,m}(\beta)$ . Using the symmetry under Hodge duality of the  $t_{n,m}$  coefficients, and renaming variables, one gets

$$\begin{aligned} \sum_{n,m=0}^d (-)^{n+m} n t_{n,m}(\beta) &= \sum_{n,m=0}^d (-)^{n+m} n t_{d-n,d-m}(\beta) = \sum_{n,m=0}^d (-)^{n+m} (d-n) t_{n,m}(\beta) \\ &= d \chi(\mathcal{M}) - \sum_{n,m=0}^d (-)^{n+m} n t_{n,m}(\beta) \end{aligned} \quad (\text{C.19})$$

proving the identity

$$\sum_{n,m=0}^d (-)^{n+m} n t_{n,m}(\beta) = \sum_{n=0}^d (-)^n n \operatorname{ind}(\Omega^{n,0}, \bar{\partial}) = \frac{d}{2} \chi(\mathcal{M}) . \quad (\text{C.20})$$

As a final discussion on topological quantities, we investigate now the identification between non-propagating top forms and the analytic Ray-Singer torsion. The effective action for a  $(p, d-1)$ -form can be written from (4.19) as

$$\begin{aligned} Z_{p,d-1}^{A \text{ top}} &= \int_0^\infty \frac{d\beta}{\beta} \sum_{n=0}^p \sum_{m=0}^{d-1} (-)^{n+m} (n+1)(m+1) t_{p-n,d-1-m} \\ &= \int_0^\infty \frac{d\beta}{\beta} \sum_{n=0}^p \sum_{m=0}^d (-)^{n+m+1} (n+1)m t_{d-p+n,m} . \end{aligned} \quad (\text{C.21})$$

We focus now on the part

$$\int_0^\infty \frac{d\beta}{\beta} \sum_{m=0}^d (-)^m m t_{d-p+n,m} .$$

In order to define the torsion [40], we recall that our hamiltonian coincides, up to a sign, with the Dolbeault laplacian and we have  $H = (\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial})$ . We shall denote with  $\lambda_n$  the eigenvalues of  $H$ , that are non-negative, and define the corresponding zeta function as

$$\zeta_{p,q}(s) = \sum_{\lambda_n > 0} \lambda_n^{-s} , \quad (\text{C.22})$$

where the subscript reminds that the hamiltonian is acting on  $(p, q)$ -forms. By taking the derivative with respect to  $s$  one can perform several manipulations

$$\zeta'_{p,q}(0) = - \sum_{\lambda_n > 0} \ln(\lambda_n) = - \operatorname{Tr}'_{\mathcal{H}_{p,q}} \ln H = \int_0^\infty \frac{d\beta}{\beta} \operatorname{Tr}'_{\mathcal{H}_{p,q}} [e^{-\beta H}] = \int_0^\infty \frac{d\beta}{\beta} t_{p,q}(\beta) . \quad (\text{C.23})$$

In the last step we rewrote the logarithm as a Schwinger proper time integral. The prime over the trace dictates to subtract zero mode contributions, that are present only if one considers a compact manifold. Viewing the formula as a field theory effective action, this

subtraction corresponds to regulating the infrared divergences due to massless fields and related to the large  $\beta$  region. At this stage we can introduce the Ray-Singer analytic  $\bar{\partial}$ -torsion

$$\ln T_p(\mathcal{M}) = \frac{1}{2} \sum_{q=0}^d (-)^q q \zeta'_{p,q}(0) , \quad (\text{C.24})$$

and we readily recognize its relation with top forms

$$Z_{p,d-1}^{A \text{ top}} = 2 \sum_{n=0}^p (-)^{n+1} (n+1) \ln T_{d-p+n}(\mathcal{M}) . \quad (\text{C.25})$$

An analogous identification arises for ordinary differential forms [41].

## References

- [1] N. Marcus and S. Yankielowicz, “The topological B model as a twisted spinning particle,” Nucl. Phys. B **432** (1994) 225 [hep-th/9408116].
- [2] N. Marcus, “Kähler spinning particles,” Nucl. Phys. B **439** (1995) 583 [hep-th/9409175].
- [3] F. Bastianelli and R. Bonezzi, “U(N) spinning particles and higher spin equations on complex manifolds,” JHEP **0903** (2009) 063 [arXiv:0901.2311 [hep-th]].
- [4] C. Iazeolla, E. Sezgin and P. Sundell, “Real forms of complex higher spin field equations and new exact solutions,” Nucl. Phys. B **791** (2008) 231 [arXiv:0706.2983 [hep-th]].
- [5] F. Bastianelli and R. Bonezzi, “Quantum theory of massless (p,0)-forms,” JHEP **1109** (2011) 018 [arXiv:1107.3661 [hep-th]].
- [6] F. Bastianelli, P. Benincasa and S. Giombi, “Worldline approach to vector and antisymmetric tensor fields,” JHEP **0504** (2005) 010 [hep-th/0503155].
- [7] F. Bastianelli, P. Benincasa and S. Giombi, “Worldline approach to vector and antisymmetric tensor fields. II,” JHEP **0510** (2005) 114 [hep-th/0510010].
- [8] F. Bastianelli and A. Zirotti, “Worldline formalism in a gravitational background,” Nucl. Phys. B **642** (2002) 372 [hep-th/0205182].
- [9] F. Bastianelli, O. Corradini and A. Zirotti, “Dimensional regularization for N=1 supersymmetric sigma models and the worldline formalism,” Phys. Rev. D **67** (2003) 104009 [hep-th/0211134].
- [10] F. Bastianelli, O. Corradini and A. Zirotti, JHEP **0401** (2004) 023 [hep-th/0312064].
- [11] D. Francia and A. Sagnotti, “Free geometric equations for higher spins,” Phys. Lett. B **543** (2002) 303 [hep-th/0207002].
- [12] D. Francia and A. Sagnotti, “On the geometry of higher spin gauge fields,” Class. Quant. Grav. **20** (2003) S473 [hep-th/0212185].
- [13] D. Francia, “String theory triplets and higher-spin curvatures,” Phys. Lett. B **690** (2010) 90 [arXiv:1001.5003 [hep-th]].
- [14] C. Fronsdal, “Massless Fields with Integer Spin,” Phys. Rev. D **18** (1978) 3624.
- [15] F. Bastianelli, O. Corradini and E. Latini, “Spinning particles and higher spin fields on (A)dS backgrounds,” JHEP **0811**, 054 (2008) [arXiv:0810.0188 [hep-th]].

- [16] S. M. Kuzenko and Z. V. Yarevskaya, “Conformal Invariance, N-extended Supersymmetry and Massless Spinning Particles in Anti-de Sitter Space,” *Mod. Phys. Lett. A* **11** (1996) 1653 [arXiv:hep-th/9512115].
- [17] D. Cherney, E. Latini and A. Waldron, “(p,q)-form Kahler Electromagnetism,” *Phys. Lett. B* **674** (2009) 316 [arXiv:0901.3788 [hep-th]].
- [18] F. Bastianelli, O. Corradini and A. Waldron, “Detours and Paths: BRST Complexes and Worldline Formalism,” *JHEP* **0905**, 017 (2009) [arXiv:0902.0530 [hep-th]].
- [19] D. Cherney, E. Latini and A. Waldron, “BRST Detour Quantization,” *J. Math. Phys.* **51** (2010) 062302 [arXiv:0906.4814 [hep-th]].
- [20] D. Cherney, E. Latini and A. Waldron, “Generalized Einstein Operator Generating Functions,” *Phys. Lett. B* **682** (2010) 472 [arXiv:0909.4578 [hep-th]].
- [21] E. Witten, “Constraints on supersymmetry breaking,” *Nucl. Phys. B* **202** (1982) 253.
- [22] E. Witten, “Supersymmetry and Morse theory,” *J. Diff. Geom.* **17** (1982) 661.
- [23] L. Alvarez-Gaume, “Supersymmetry and the Atiyah-Singer index theorem,” *Commun. Math. Phys.* **90** (1983) 161.
- [24] J. M. Figueroa-O’Farrill, C. Kohl and B. J. Spence, “Supersymmetry and the cohomology of (hyper)Kahler manifolds,” *Nucl. Phys. B* **503** (1997) 614 [hep-th/9705161].
- [25] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, “A New supersymmetric index,” *Nucl. Phys. B* **386** (1992) 405 [hep-th/9204102].
- [26] F. Bastianelli, O. Corradini and E. Latini, “Higher spin fields from a worldline perspective,” *JHEP* **0702**, 072 (2007) [hep-th/0701055].
- [27] F. Bastianelli, “The path integral for a particle in curved spaces and Weyl anomalies,” *Nucl. Phys. B* **376** (1992) 113 [hep-th/9112035].
- [28] F. Bastianelli and P. van Nieuwenhuizen, “Trace anomalies from quantum mechanics,” *Nucl. Phys. B* **389** (1993) 53 [hep-th/9208059].
- [29] K. Higashijima and M. Nitta, “Kähler normal coordinate expansion in supersymmetric theories,” *Prog. Theor. Phys.* **105** (2001) 243 [arXiv:hep-th/0006027].
- [30] F. Bastianelli and P. van Nieuwenhuizen, “Path integrals and anomalies in curved space,” Cambridge University Press, Cambridge UK (2006).
- [31] J. De Boer, B. Peeters, K. Skenderis and P. Van Nieuwenhuizen, “Loop calculations in quantum mechanical nonlinear sigma models,” *Nucl. Phys. B* **446** (1995) 211 [arXiv:hep-th/9504097];
- [32] J. de Boer, B. Peeters, K. Skenderis and P. van Nieuwenhuizen, “Loop calculations in quantum-mechanical non-linear sigma models sigma models with fermions and applications to anomalies,” *Nucl. Phys. B* **459** (1996) 631 [arXiv:hep-th/9509158].
- [33] F. Bastianelli, K. Schalm and P. van Nieuwenhuizen, “Mode regularization, time slicing, Weyl ordering and phase space path integrals for quantum mechanical nonlinear sigma models,” *Phys. Rev. D* **58** (1998) 044002 [hep-th/9801105].
- [34] F. Bastianelli and O. Corradini, “On mode regularization of the configuration space path integral in curved space,” *Phys. Rev. D* **60** (1999) 044014 [hep-th/9810119].
- [35] R. Bonezzi and M. Falconi, “Mode Regularization for N=1,2 SUSY Sigma Model,” *JHEP*

- 0810** (2008) 019 [arXiv:0807.2276 [hep-th]].
- [36] H. Kleinert and A. Chervyakov, “Reparametrization invariance of path integrals,” *Phys. Lett. B* **464** (1999) 257 [hep-th/9906156].
  - [37] F. Bastianelli, O. Corradini and P. van Nieuwenhuizen, “Dimensional regularization of the path integral in curved space on an infinite time interval,” *Phys. Lett. B* **490** (2000) 154 [hep-th/0007105].
  - [38] F. Bastianelli, O. Corradini and P. van Nieuwenhuizen, “Dimensional regularization of nonlinear sigma models on a finite time interval,” *Phys. Lett. B* **494**, 161 (2000) [hep-th/0008045].
  - [39] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, “Extended SUSY quantum mechanics: transition amplitudes and path integrals,” *JHEP* **1106** (2011) 023 [arXiv:1103.3993 [hep-th]].
  - [40] D. B. Ray and I. M. Singer, “Analytic torsion for complex manifolds,” *Annals Math.* **98** (1973) 154.
  - [41] A. S. Schwarz and Y. S. Tyupkin, “Quantization Of Antisymmetric Tensors And Ray-Singer Torsion,” *Nucl. Phys. B* **242**, 436 (1984).
  - [42] M. Nakahara, “Geometry, Topology and Physics,” Institute of Physics Publishing, Bristol and Philadelphia (2003).